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Element Free Galerkin (EFG) sensitivity study in structural analysis

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Abstract. The present study shows a parametric analysis of the meshfree method, Element Free Galerkin (EFG), on the elastic analysis of a cantilever beam. This study allows us to determine the best convergence conditions of the solutions varying characteristic. EFG is based on the construction of Moving Least Squares (MLS) approximations using the weighted residual method on the weak formulation, with MLS form functions as the same weighting functions. We consider the parameters of the method such as the order of the basic functions of MLS functions, the size of the support domain of the local MLS functions and the density of Gauss points against errors calculated according to the L_2 norm and processing time. It is shown that by increasing the order of basic functions it is possible to obtain more precise results, however, a larger support diameter and Gauss points higher density are required in order to stabilize the solution, considerably increasing processing times. Therefore, it is only advisable to use high-order base functions when the precision in the results is the priority and a high computational resource is available.

1. Introduction

As FEM (The Finite Element Method) has proven its efficiency and capacity in all areas of engineering, from a practical point of view, its use on real models is always associated with numerous man-hours dedicated to the generation of meshes. Geometric irregularities in mechanical engineering parts usually represent great difficulties to obtain proper geometric discretizations as required by FEM. Therefore, meshfree methods have arisen, which try not to base the approximation on meshes, but on independent points that do not require a pre-defined interconnection [1].

Meshfree methods would allow us to solve problems with large deformations, and complex geometries, of topological optimization without the great difficulties that afflict finite element method. All this, thanks to the fact that the approach is based on nodes that have associated support domains, see Fig. 1. These nodes do not require a pre-defined connectivity within of the general domain [2], which implies that the approach functions do not require a mesh.

On the other hand, not requiring a mesh to achieve a good approximation results, in a greater difficulty when imposing border conditions [3]. This aspect has been overcome by applying more elaborate methods to guarantee the imposition of values at the borders. Methods such as penalization and lagrange multipliers are widely used with meshfree methods achieving good results [4][5][6].



The approximation functions by Moving Least Squares (MLS) to allow local discretization, with arbitrarily located points, has become one of the most outstanding options for use in the meshfree methods. For that reason there are currently a good number of researchers developing and demonstrating the advantages of its use. Examples of this can be observed in the works developed by E. Shivanian [7], where you test a methodology based on local approaches with MLS functions (*"Meshless Local Petrov-Galerkin"*, MLPG) to solve wave problems in three dimensions, by Hongping R. [8] and F.X. Sun [9] demonstrating the convergence of MLS functions in the study of problems in two and n dimensions, respectively, by Mehdi D. and Vahind M. [10] that use the generalized formulation of MLS (GMLS) to simulate problems of scalar fields in the theory of quantum physics and by E. Dabboura [11] who demonstrates the versatility of MLS functions by solving the nonlinear equation of *"Kuramoto-Sivashinsky"*, among other works that are found in the literature ([12], [13], [14], [15], [16]).

One of the most used and robust mesh free methods is the Element Free Galerking (EFG) [17]. Which seeks to solve the weak formulation of a partial differential equation, using the same MLS functions as approximation functions and as weighting functions. EFG, although is one of the most used methods, uses of several parameters that must be adjusted to ensure good convergence. The present work contains a sensitivity study of these parameters of the method.

2. Methodology

2.1. Support domain

The general domain will be discretized using randomly located points, which will be associated with a support domain with the surrounding nodes. These domains will be constructed with simple geometric forms (Circles or rectangles, see Fig. 1) and from their contributions it will be possible to construct the approximation on the general domain.

At the moment of generating the support domains, we have to think how big it should be. If we can locate as many nodes as terms are in the base function for the approximation ($p(x) = [1 \ x \ y]$), we will achieve a better approximation that satisfies the *Kronecker* delta condition. However, the idea of MLS approach is to use randomly distributed nodes so that it can not always be predicted how many nodes will there be in support domains. For this reason the *Kronecker* Delta property is not always met. It is necessary to emphasize that very small support domains can cause the momentum matrix $M(x)$ (See Eq. 5) to be singular; and very large domains require a great computational resources.

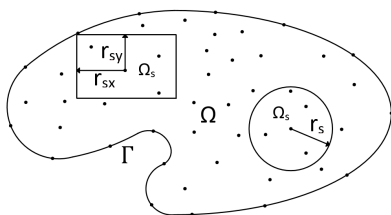


Figure 1. Support domain circular and rectangular mean in a general domain limited by boundary Γ .

2.2. MLS approximation

We want to approximate a function $u(x)$ in a domain Ω to use MLS approximation functions, as defined in Eq. (1).

$$u(x) = p^t(x)a(x) \quad (1)$$

where,

- $u(x)$ is the function to approximate.
- $p^t(x) = [p_1(x) \ p_2(x) \ p_3(x) \ \dots \ p_m(x)]$: are functions created from polynomial bases, defined for a point $x^t = [x, y]$, with m equal to the number of terms that will depend on the degree of the polynomial constructed with the pascal triangle.
- $a(x)$ is the vector of constant coefficients of the base functions.

2.3. ϕ function.

To obtain the value of the coefficients $a(x)$ it is necessary to define the functional in Eq. (2) [18].

$$J = W(x - x_i)[P^T(x)a(x) - u^*]^2 \quad (2)$$

The function $W(x - x_i)$ is a diagonal matrix of weight functions(See Eq. (3)).

$$W_{kj}(x - x_i) = \delta_{kj}W_k(x - x_i) \quad (3)$$

By minimizing the functional $J(x)$, we get:

$$P^T(x)W(x - x_i)P(x)a(x) = W(x - x_i)P(x)u^* \quad (4)$$

Which can be represented as the linear system in Eq. (5):

$$M(x)a(x) = B(x)u^* \quad (5)$$

Where,

- $M(x) = P^T(x)W(x - x_i)P(x)$: is defined as the moment matrix and has dimensions $[m \times m]$.
- $B(x) = P^T(x)W(x - x_i)$: is a rectangular matrix.

Finally it is possible to rewrite Eq. (1) as shown in Eq. (6).

$$u(x) = \Phi(x)^T u^* \quad (6)$$

2.4. EFG formulation

The equation in partial derivatives describing the elastic behavior of a 2D body and its boundary conditions are posed as follows [19]:

$$L^T \sigma + b = 0 \quad (7)$$

$$u_i = \bar{u}_i \quad \text{en } \Gamma_u \quad (8)$$

$$\sigma_{ij}n_j = \bar{t}_i \quad \text{en } \Gamma_t \quad (9)$$

Where, L is a differential matrix operator [18] like the form in Eq.(10).

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (10)$$

Using Galerkin's method with weighted residues [20] on Eq. (7) and imposing boundary conditions through the penalty method [21], we obtain :

Table 1. Properties of the study case.

Propiedad	Valor
Young's modulus (E)	30e6 Pa
Poisson's ratio (ν)	0.3
Load (P)	-1000 N
Length (L)	48 m
Heigh (D)	12 m
Thickness (t)	12 m

$$\int_{\Phi} (L\Phi)^t D(L\Phi)u \cdot d\Omega + \int_{\Gamma_u} (\Phi^t \alpha \Phi)u \cdot d\Gamma = \int_{\Omega} \Phi b \cdot d\Omega - \int_{\Gamma_t} \Phi T \cdot d\Gamma + \int_{\Gamma_u} \Phi \alpha \bar{u} \cdot d\Gamma \quad (11)$$

Solving the integrals of the Eq. (11) we obtain a system of linear equations:

$$(K + K^\alpha)u = F + F^\alpha \quad (12)$$

Where,

$$K = \int_{\Phi} (L\Phi)^t D(L\Phi)u \cdot d\Omega \quad (13)$$

$$K^\alpha = \int_{\Gamma_u} (\Phi^t \alpha \Phi)u \cdot d\Gamma \quad (14)$$

$$F = \int_{\Omega} \Phi b \cdot d\Omega - \int_{\Gamma_t} \Phi T \cdot d\Gamma \quad (15)$$

$$F^\alpha = \int_{\Gamma_u} \Phi \alpha \bar{u} \cdot d\Gamma \quad (16)$$

3. Numerical results and discussions

The case of a "Cantilever beam" [20, 22, 23], which refers to a structural element embedded in one of its ends with a load perpendicular to the axis of the element (See Fig. 2), will be studied. The mechanical properties of the model and the load conditions are defined in Table 1.

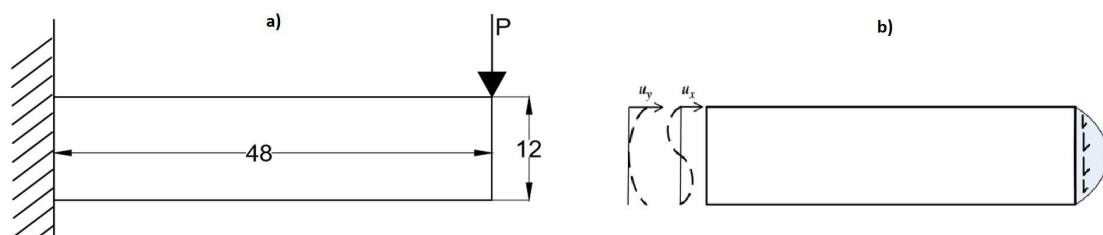


Figure 2. a) Cantilever beam model. b) Scheme of natural and essential boundary conditions.

3.1. Boundary Condition.

For Cantilever beams the exact solution is the Timoshenko and Goodier beam, as found in the literature [22], we will take specified displacements at $x = 0$, which can be written as Eqs. (17)-(18).

$$u(x, y) = -\frac{Py}{6EI} \left[(6L - 3x)x + (2 + \nu) \left(y^2 - \frac{D^2}{4} \right) \right] \quad (17)$$

$$v(x, y) = -\frac{P}{6EI} \left[3\nu y^2(L - x) + (4 + 5\nu) \frac{D^2 x}{4} + (3L - x)x^2 \right] \quad (18)$$

For the natural boundary condition the load P located in the free ending ($x=48$) will generate a shear stress distributed in a parabolic form along the beam, according to the Eq. (19).

$$\tau_{xy} = -\frac{P}{2I} \left(\frac{D^2}{4} - y^2 \right) \quad (19)$$

Finally, both boundary conditions are shown in the Fig. 2.

3.2. Analytical solution

The solution proposed by Timoshenko and Goodier [23] is given by Eqs. (17)-(21) for displacements, shear stress and normal stress.

$$\sigma_x = -\frac{P(L - x)y}{I} \quad (20)$$

$$\sigma_y = 0 \quad (21)$$

The analytical results given by Timoshenko and Goodier can be plotted as in Fig. 3.

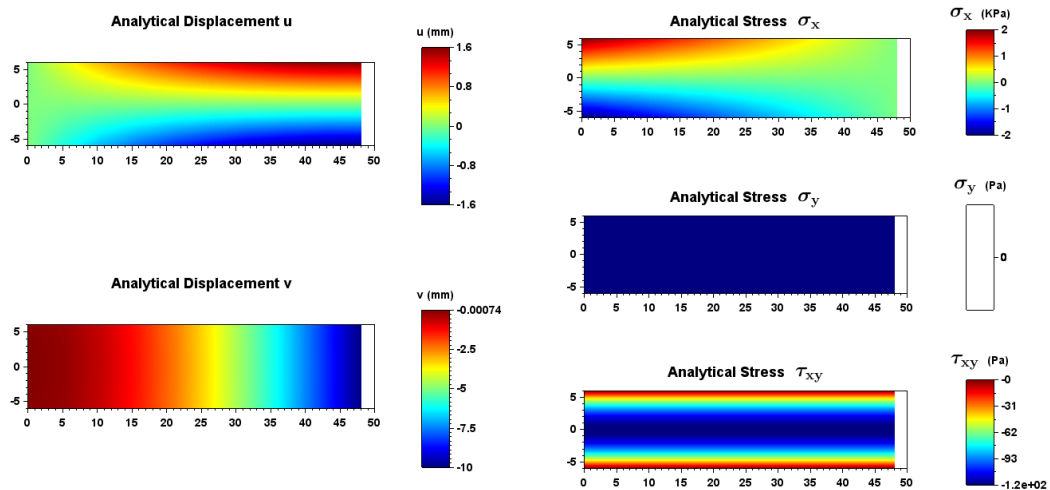


Figure 3. The Cantilever beam analytical solution.

3.3. Convergence analysis.

Having defined load conditions for the model and an given an analytical solution, we proceed to perform a sensitivity analysis on the parameters of EFG method, to define configurations that will give good solutions without requiring excessive computational resources (measured as processing time). In this sensitivity analysis the parameters to be studied are:

- **Order of base functions $p(x)$:** Increasing the order of polynomial in the base functions of EFG method should increases the speed of convergence and the accuracy of EFG [24], which is why it is proposed to study their effect on the quality of the approximation.
- **Size of Support domains:** Rectangular regions whose size is measured according to the percentage of nodes that are covered when taken as support domains. For example, in a domain with 200 nodes, a support domain of 10% means that on average the support domains cover 20 nodes.
- **Number of Gauss points:** The number of Gauss points will be an important factor in solving the integrals of the EFG formulation. It will be shown how many Gauss points are required to achieve a good convergence and how to adjust it depending on the support domain size and the order of polynomial base functions. Experiments were performed using up to 64 Gauss points within in each cell, and varying this density.
- **Processing time and error:** Processing time and the relative and absolute error will be taken as control parameters to test solutions in order to find solutions not so expensive in processor time but also ones that generate acceptable errors. Error will be measured in comparison with the analytical response of the model, and calculated according to the energy standard, as being one of the most used norms in elasticity [25], see equation 22.

$$\epsilon_{\%} = \frac{\left(\int (\sigma_{exac} - \sigma_{efg})D^{-1}(\sigma_{exac} - \sigma_{efg})dA\right)^{1/2}}{\int \sigma_{exac}dA} \quad (22)$$

The graphs 4, 5, 6 and 7 are the results obtained by varying each of the mentioned parameters in a sample space of 128 iterations. The vertical axis of the left graphs details the error percentage, the vertical axis of the right graphs sets the processing time, the horizontal axis alludes to the number of Gauss points, the colors of the lines represent the size average percentage of media domains and each graph shows the results for different order polynomials base function. The studies were performed with uniform's distributed nodes using a 10x10 rectangular grid through the beam.

It is possible to appreciate how, as the polynomial basis functions order increases, the response improves, however, a good density of Gaussian points and large support domains are required to stabilize it, on the other hand, this causes a considerable increase in the processing time. Given this, it is only recommended to use higher order functions when higher precision is required and we have great computational resources.

From the graphs, it is possible to recover the support domains and the number of Gauss points that produced the least processing times with admissible errors, these best configurations are found in the table 2.

Table 2. Best results for the second order polynomial base funciont

Order	S. Domain	# Gauss Point	Time (s)	Error
1	54%	2304	22, 94	0.485%
2	36%	2304	18, 86	0.392%
3	54%	2304	32, 58	0.215%
4	107%	3106	107, 79	0.099%

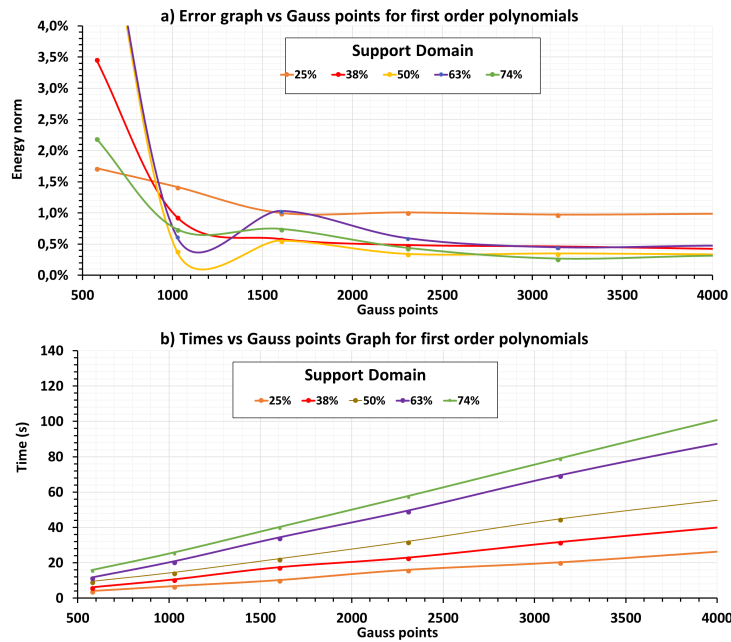


Figure 4. Results of elastic solution to the Cantilever beam problem using EFG with first order polynomials basis functions.

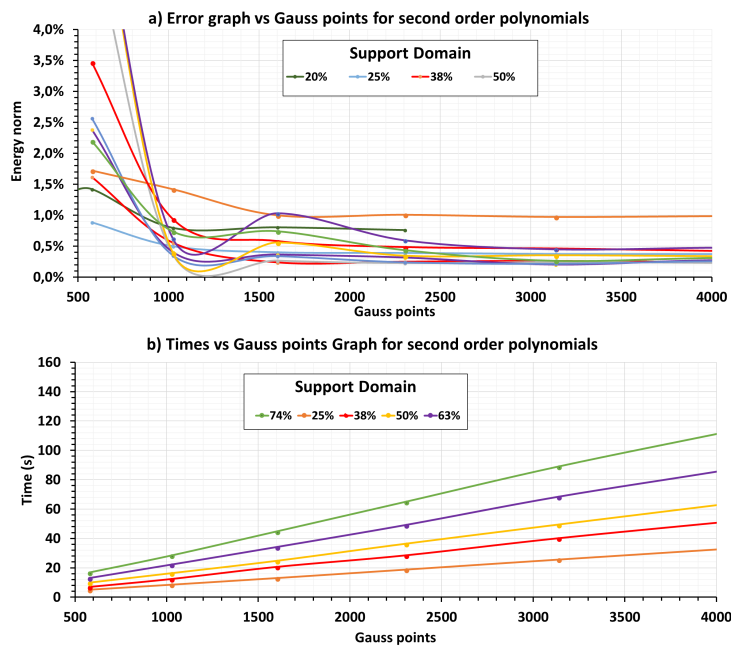


Figure 5. Results of elastic solution to the Cantilever beam problem using EFG with second order polynomials basis functions.

3.4. EFG Solution

From the convergence analysis of the previous section it can be seen that the best configuration, in table 2, was for second order polynomials basis function with rectangular support domains. The results of this configuration can be seen in Fig. 8. It is worthwhile to clarify that the σ_y

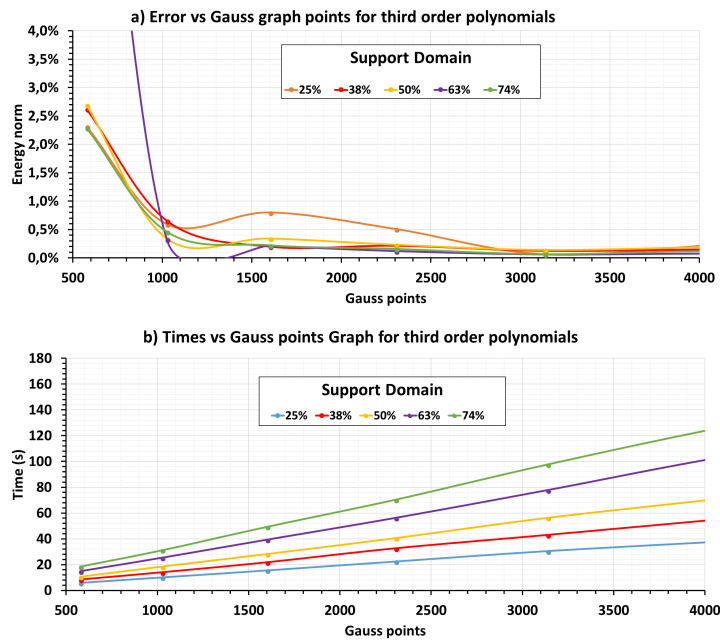


Figure 6. Results of elastic solution to the Cantilever beam problem using EFG with third order polynomials basis functions.

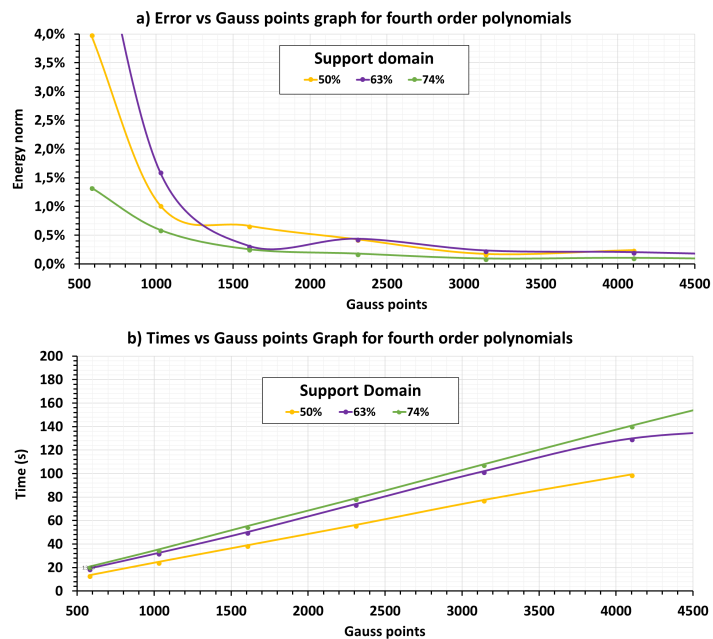


Figure 7. Results of elastic solution to the Cantilever beam problem using EFG with fourth order polynomials basis functions.

absolute error obtained is below 0.0005 when put in the scale of loads applied.

As a last test it is possible to compare the results obtained by EFG in Fig. 3 with the results of the analytical solution of the Fig. 8 and we can appreciate the great similarity between them, as well as the continuity in the solutions. So finally EFG achieved a valid solution to the

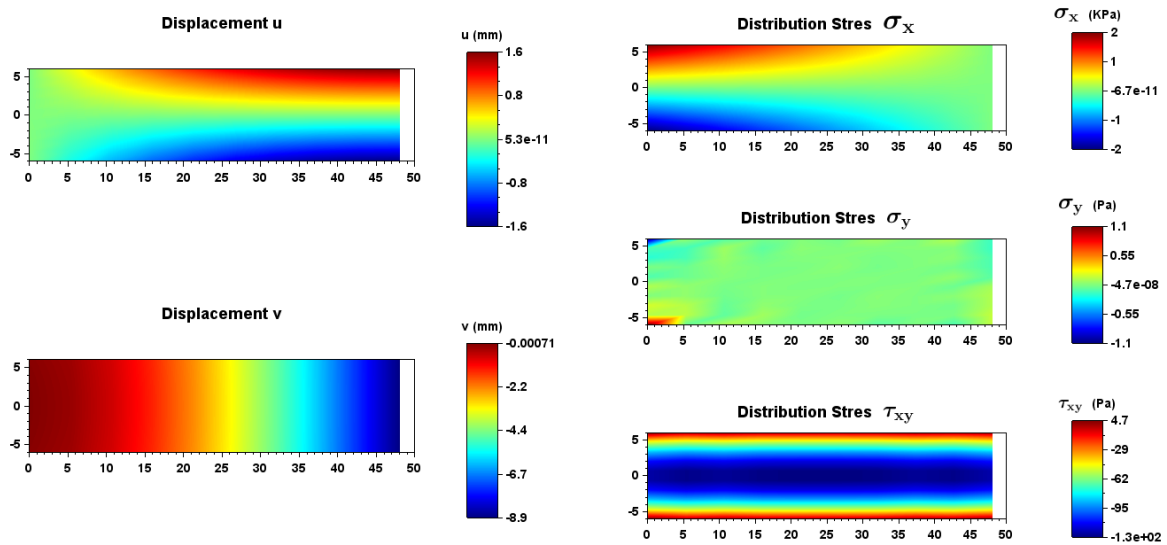


Figure 8. Elastic solution for Cantilever beam using EFG.

Cantilever beam.

4. Conclusions

The EFG method can improve its convergence by varying parameters such as the Gauss points density, the size of support domains, and the polynomial basis functions order ($P(x)$) of the functions Φ . The higher the order of polynomial basis functions, the more Gaussian points and larger support domains will be required to stabilize the method, resulting in longer processing time.

It is advisable to work with initial support diameters of at least 38% of the total nodes on average and with first order polynomials basis functions, in order to achieve a rapid convergence of the problem. For example if we use a 1000 node domain we could to start, in average, with support domain of 380 nodes.

For the number of Gauss points, it is recommended to use at least 8 times the number of variables (In mechanical structural cases u and v) to solve when using first and second order polynomials basis functions and for third and fourth order polynomials basis function at least 15 times the number of variables.

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