



## Equivalence of categories of simple Lie algebras in positive characteristic

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*Received : February 2022. Accepted : March 2023*

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### Abstract

*In this paper we first study some properties of the finite-dimensional simple restricted Lie algebras. In the literature is found a one-to-one correspondence between them and finite-dimensional simple Lie algebras over a field of positive characteristic. Motivated by this fact, we give a one-to-one correspondence between their morphisms, which allow us to conclude that such categories are equivalent.*

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**Subclass [2010]:** *17B50 (primary) 17B20, 18A22, 18A23 (secondary).*

**Keywords:** *Restricted Lie algebras, restricted Simple Lie algebras, Simple restricted Lie algebras, Equivalence of categories*

## 1. Introduction

The finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero are well known. A better description can be obtained for instance in [4]. They are divided in four infinite families: the special linear algebra  $A_n = \mathfrak{sl}(n+1)$  for  $n \geq 1$ , the special orthogonal algebra of odd rank  $B_n = \mathfrak{so}(2n+1)$  for  $n \geq 2$ , the symplectic algebra  $C_n = \mathfrak{sp}(2n)$  for  $n \geq 3$  and the special orthogonal algebra of even rank  $D_n = \mathfrak{so}(2n)$  for  $n \geq 4$ ; and five exceptional cases  $E_6$ ,  $E_7$ ,  $E_8$ ,  $G_2$  and  $F_4$ . Such algebras also can be defined over an arbitrary field of characteristic  $p > 0$ , in the following way: for a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero, one chooses a Chevalley basis of the Lie algebra, i.e., a basis of the underlying vector space of the Lie algebra such that all the structure constants are integers, and then one reduces these structure constants modulo a prime number  $p$ . These new Lie algebras are called Lie algebras of classical type and they are simple for  $p \geq 5$  (for details see [16, Ch. 4]).

However, over algebraically closed fields of characteristic  $p \geq 5$  there also exist several families of non-classical Lie algebras. Similar to Cartan's four families of finite-dimensional simple Lie algebras, there are four infinite families of Lie algebras of generalized Cartan type, namely algebras of Witt, special, Hamiltonian, or contact type. Moreover, in characteristic 5 there exists another infinite family of so-called Melikian algebras.

The classification problem of simple Lie algebras of finite dimension over an algebraically closed field  $\mathbf{F}$  of characteristic  $p > 3$  was resolved in the period between 1980 and 2008 by Block–Wilson–Premet–Strade in several of their works [1, 2, 3, 6, 7, 8, 9, 11, 12, 18, 16, 15, 14]. They give an answer to the generalized Kostrikin–Shafarevich conjecture, formulated by V. Kac in [5]. Exactly, they proved that, every finite-dimensional simple Lie algebra over  $\mathbf{F}$  with  $p > 3$  is classical (and hence restricted), a filtered Lie algebra of Cartan or Melikian type.

In characteristic  $p = 2, 3$  there are many examples of simple Lie algebra and simple restricted Lie algebras. The classification problem of these algebras still remains open. The main result for the classification problem of simple Lie algebras in characteristic 2 and 3 was obtained by S. Skryabin in [13]. He proved that any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic 2 has absolute toral rank greater than or equal to 2. In the same paper, Skryabin also proved that any finite-dimensional simple Lie algebra of absolute toral rank 1 over an algebraically

closed field of characteristic 3 is isomorphic to either  $\mathfrak{sl}(2)$  or  $\mathfrak{psl}(3)$ . In [10, Problem 1], A. Premet and H. Strade present the following problem which is still open.

**Problem 1:** *Classify all finite-dimensional simple Lie algebras of absolute toral rank two over an algebraically closed field of characteristics 2 and 3.*

In the case of absolute toral rank 2, work in progress by A. Grishkov and A. Premet was announced in [10]. Their main result states that every finite-dimensional simple Lie algebra of absolute toral rank 2 over an algebraically closed field of characteristic 2 is classical of dimension 3, 8, 14, or 26. The problem is still open in characteristic 3. Furthermore, extending Problem 1 to absolute toral rank greater than two, is a problem still open in characteristic two and three.

It is well-known that there is a one-to-one correspondence between simple restricted Lie algebras and simple Lie algebras over  $\mathbf{F}$  (cf. [19, Proposition 4.1]). However, only this fact does not imply the equivalence between such categories, for instance, the category associated to a group  $G$  has a unique object and the set of morphisms is  $G$  itself.

Motivated by the equivalence of *The classification problem* between simple Lie algebras and simple restricted Lie algebras, we prove in this paper that such categories are equivalent for any characteristic  $p > 0$ , which is the main result of this paper.

The paper is organized as follows. In Section 2 we recall the general notions of restricted Lie algebras (or Lie  $p$ -algebras), of  $p$ -envelope and minimal  $p$ -envelope of a Lie algebra; we also give some known facts. In Section 3 we define two functors between simple Lie algebras and simple restricted Lie algebras, and we prove that these functors form a pair of adjoint equivalences of categories.

## 2. Preliminaries

In this paper  $\mathbf{F}$  denotes a field of characteristic  $p > 0$  and all Lie algebras are assumed to be finite-dimensional over  $\mathbf{F}$ . We denote by  $\mathrm{ad}_L$  the adjoint representation of a Lie algebra  $L$  and by  $\mathrm{ad}_L^n(x)$  the composition  $(\mathrm{ad}_L(x))^n$  for all  $x \in L$  and  $n \in \mathbf{N}$ . We shall be interested in a particular class of Lie algebras, called restricted Lie algebras (or Lie  $p$ -algebras). Such concept was introduced by N. Jacobson [4, p. 187].

**Definition 2.1.** *Let  $L$  be a Lie algebra. A map  $[p]: L \longrightarrow L$  such that  $a \mapsto a^{[p]}$  is called  $p$ -map if*

1.  $(\lambda a)^{[p]} = \lambda^p a^{[p]}$ , for all  $\lambda \in \mathbf{F}$  and for all  $a \in L$ .
2.  $\text{ad}_L^p(a) = \text{ad}_L(a^{[p]})$ , for all  $a \in L$ .
3.  $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$ , for all  $a, b \in L$ .

where  $s_i(a, b)$  is the coefficient at  $t^{i-1}$  in the expansion of  $\text{ad}_L^{p-1}(at + b)(a)$  over the indeterminate  $t$ . A Lie algebra with a  $p$ -map is called Lie  $p$ -algebra and will be denoted by  $(L, [p])$ .

Some classical notions adapted to  $p$ -algebras follow:

**Definition 2.2.** Let  $L$  be a Lie  $p$ -algebra. A  $p$ -subalgebra ( $p$ -ideal) is a subalgebra (ideal) of  $L$  which is closed under its  $p$ -map.

A simple Lie  $p$ -algebra ( $p$ -simple Lie  $p$ -algebra) is a Lie  $p$ -algebra which does not have nonzero proper ideals ( $p$ -ideals).

In the literature a Lie  $p$ -algebra is frequently called a restricted Lie algebra. In this sense a  $p$ -simple Lie  $p$ -algebra is exactly a simple restricted Lie algebra.

Every simple Lie  $p$ -algebra is a  $p$ -simple Lie  $p$ -algebra, but its converse is not true. For instance, the Lie algebra of derivations

$$\text{Der}(\mathfrak{so}(3)) = \{(a_{ij}) \in M_3(\mathbf{F}) \mid a_{ij} = a_{ji} \text{ and } a_{11} + a_{22} + a_{33} = 0\},$$

of the special orthogonal Lie algebra  $\mathfrak{so}(3)$  over  $\mathbf{F}$  with  $\text{char}(\mathbf{F}) = 2$  has a canonical basis  $\{h_1, h_2, e_1, e_2, e_3\}$ , where  $h_1 := e_{11} + e_{22}$ ,  $h_2 := e_{22} + e_{33}$ ,  $e_1 := e_{12} + e_{21}$ ,  $e_2 := e_{13} + e_{31}$ ,  $e_3 := e_{23} + e_{32}$ . Table 2.1 exhibits inside its diagonal the 2-map and outside its Lie bracket.  $\text{Der}(\mathfrak{so}(3))$  is a 2-simple Lie 2-algebra which is not a simple Lie 2-algebra, because  $\text{span}\{e_1, e_2, e_3\}$  is an ideal of  $\text{Der}(\mathfrak{so}(3))$ .

	$h_1$	$h_2$	$e_1$	$e_2$	$e_3$
$h_1$	$h_1$	0	0	$e_2$	$e_3$
$h_2$	0	$h_2$	$e_1$	$e_2$	0
$e_1$	0	$e_1$	$h_1$	$e_3$	$e_2$
$e_2$	$e_2$	$e_2$	$e_3$	$h_1 + h_2$	$e_1$
$e_3$	$e_3$	0	$e_2$	$e_1$	$h_2$

Table 2.1: Lie bracket and 2-map on  $Der(\mathfrak{so}(3))$

**Definition 2.3.** Let  $(L_1, [p]_1)$  and  $(L_2, [p]_2)$  be two Lie  $p$ -algebras. A  $p$ -homomorphism  $f : L_1 \rightarrow L_2$  is a Lie homomorphism such that  $f(a^{[p]_1}) = f(a)^{[p]_2}$  for all  $a \in L_1$ .

For a category  $\mathcal{A}$  (Lie algebras or Lie  $p$ -algebras),  $\mathcal{A}^s$  (respectively  $\mathcal{A}^{ss}$ ) will denote the full subcategory of  $\mathcal{A}$  consisting of simple objects (respectively semisimple) and  $\mathcal{A}^{[p]}$  the full subcategory of  $\mathcal{A}$  whose objects have a  $p$ -structure.

**Remark 2.4.** Denote by **Lie** the category of finite-dimensional Lie algebras. From above-mentioned, there exists a difference between the categories  $(\mathbf{Lie}^{[p]})^s$  and  $(\mathbf{Lie}^s)^{[p]}$ . However, the following relation between their objects holds:

$$\text{Obj}(\mathbf{Lie}^s) \cap \text{Obj}(\mathbf{Lie}^{[p]})^s = \text{Obj}(\mathbf{Lie}^s)^{[p]}.$$

For a subset  $S$  of a Lie  $p$ -algebra  $(L, [p])$ , the  $p$ -subalgebra generated by  $S$  in  $L$  is  $S_p := \bigcap_{i \in I} H_i$ , where  $\{H_i \mid i \in I\}$  is the family of all  $p$ -subalgebras which contain  $S$ . Such  $S_p$  is the *smallest*  $p$ -subalgebra of  $L$  containing  $S$ .

**Proposition 2.5.** [17, p. 66] Let  $L$  be a Lie  $p$ -algebra and let  $M$  be a Lie subalgebra of  $L$  and  $\{e_1, e_2, \dots, e_n\}$  a basis for  $M$ . Then

1.  $M_p = \text{span}_{\mathbf{F}} \{e_i^{[p]^j} \mid i = 1, 2, \dots, n; j \in \mathbf{Z}_{\geq 0}\}$ .
2.  $[L, M_p] = [L, M]$  and  $[M_p, M_p] = [M, M]$ .
3. If  $I$  is an ideal of  $L$ , then  $I_p$  is a  $p$ -ideal of  $L$ .

**Lemma 2.6.** Let  $L$  be a Lie  $p$ -algebra and let  $M$  be a Lie subalgebra of  $L$ . If  $I$  is an ideal of  $M$ , then  $I$  is an ideal of  $M_p$ .

**Proof.** Let  $x \in I$  and  $y \in M_p$  be arbitrary elements. Suppose that  $\{e_1, e_2, \dots, e_n\}$  is a basis of  $M$ . From Proposition 2.5

$$y = \sum_{i,j} \alpha_{ij} e_i^{[p]^j}.$$

Therefore,

$$[x, y] = \sum_{i,j} \alpha_{ij} [x, e_i^{[p]^j}] = - \sum_{i,j} \alpha_{ij} \text{ad}_L(e_i^{[p]^j})(x) = - \sum_{i,j} \alpha_{ij} \text{ad}_L^{p^j}(e_i)(x) \in I.$$

□

**Lemma 2.7.** *If  $I$  is a nonzero ideal of  $L \in (\mathbf{Lie}^{[p]})^s$ , then  $I_p = L$ . Furthermore,  $[L, L] \subseteq J$  for any nonzero ideal  $J$  of  $I$ .*

**Proof.** Since  $I_p$  is a  $p$ -ideal (Proposition 2.5) and  $L \in (\mathbf{Lie}^{[p]})^s$  we deduce that  $I_p = L$ . Now, consider a nonzero ideal  $J$  of  $I$ . From Lemma 2.6 follows that  $J$  is an ideal of  $I_p = L$ . Proposition 2.5 now implies that  $J_p$  is a  $p$ -ideal of  $L$ , which yields  $J_p = L$  because  $L \in (\mathbf{Lie}^{[p]})^s$ . Finally, the same proposition gives  $[L, L] = [L, J_p] = [L, J] \subseteq J$ . □

The following well-known result (see [19, Proposition 4.1]) is a consequence of the previous lemma.

**Corollary 2.8.** *If  $L \in (\mathbf{Lie}^{[p]})^s$ , then  $[L, L]_p = L$  and  $[L, L] \in \mathbf{Lie}^s$ .*

**Proof.** Let us consider  $I = [L, L]$ . If  $J$  is a nonzero ideal of  $I$  we have  $I = [L, L] \subseteq J$ . □

The next concept is the main tool to define our functor from  $\mathbf{Lie}^s$  to  $(\mathbf{Lie}^{[p]})^s$ .

**Definition 2.9.** [17, pp. 92 – 94] *Let  $L$  be a Lie algebra.*

1. A triple  $(G, [p], \iota)$  is called a  **$p$ -envelope** of  $L$  if  $(G, [p])$  is a Lie  $p$ -algebra and  $\iota: L \rightarrow G$  is a Lie algebra monomorphism such that  $\iota(L)_p = G$ .
2. A  $p$ -envelope  $(G, [p], \iota)$  of  $L$  is called **minimal** if its dimension is minimal among the dimensions of all  $p$ -envelopes of  $L$ .

The following proposition is well-known (see for example, [19, Proposition 4.1])

**Proposition 2.10.** *If  $L \in \mathbf{Lie}^s$ , then  $\text{ad}_L(L)_p \in (\mathbf{Lie}^{[p]})^s$  is the minimal  $p$ -envelope of  $L$  with the monomorphism  $\text{ad}_L : L \rightarrow \text{ad}_L(L)$ .*

### 3. Functorial approach

In the literature (see [3, 19]) is possible to find a one-to-one correspondence of objects between  $(\mathbf{Lie}^{[p]})^s$  and  $\mathbf{Lie}^s$ . Thus the classification of the finite-dimensional  $p$ -simple Lie  $p$ -algebras is equivalent to classification of the finite-dimensional simple Lie algebras. We will show that the categories  $(\mathbf{Lie}^{[p]})^s$  and  $\mathbf{Lie}^s$  are equivalent.

To this end, let us start with the functor from  $p$ -simple Lie  $p$ -algebras to simple Lie algebras

**Proposition 3.1.** *There exists a dense and faithful covariant functor  $\mathcal{F} : (\mathbf{Lie}^{[p]})^s \rightarrow \mathbf{Lie}^s$  given by*

$$\mathcal{F}(L) := [L, L], \text{ for all } L \in \text{Obj}(\mathbf{Lie}^{[p]})^s,$$

$$\mathcal{F}(f) := f|_{[L, L]}, \text{ for all } f \in \text{Hom}_{(\mathbf{Lie}^{[p]})^s}(L, L').$$

**Proof.** It is easy to check from Corollary 2.8 that  $\mathcal{F}$  is a covariant functor. Now, consider  $f, g \in \text{Hom}_{(\mathbf{Lie}^{[p]})^s}(L, L')$  such that  $\mathcal{F}(f) = \mathcal{F}(g)$ , so  $f|_{[L, L]} = g|_{[L, L]}$ , which implies that  $f([x, y]^{[p]_2^n}) = g([x, y]^{[p]_2^n})$ , for all  $x, y \in L$  and  $n \in \mathbf{Z}_+$ . Since  $f, g$  are  $p$ -homomorphisms we have  $f([x, y]^{[p]_1^n}) = g([x, y]^{[p]_1^n})$ . It follows from Proposition 2.5 that  $f|_{[L, L]_p} = g|_{[L, L]_p}$  and  $[L, L]_p$  is a  $p$ -ideal of  $L$ . Therefore  $L = [L, L]_p$  and  $f = g$ .

We proceed to show that  $\mathcal{F}$  is dense. From Proposition 2.10, for every  $L \in \mathbf{Lie}^s$  its minimal  $p$ -envelope  $G = \text{ad}_L(L)_p$  of  $L$  is an object in  $(\mathbf{Lie}^{[p]})^s$ . Furthermore, it follows from Proposition 2.5 and the simplicity of  $L$  that

$$\mathcal{F}(G) = [\text{ad}_L(L)_p, \text{ad}_L(L)_p] = [\text{ad}_L(L), \text{ad}_L(L)] = \text{ad}(L) \cong L.$$

□

The remainder of this section will be devoted to the construction of a covariant functor  $\mathcal{G} : \mathbf{Lie}^s \rightarrow (\mathbf{Lie}^{[p]})^s$  which is an equivalence of categories.

Let  $\mathcal{L}$  be the category either  $\mathbf{Lie}^{ss}$  or  $\mathbf{Lie}^s$ . Since  $\text{ad}_L : L \rightarrow \text{ad}_L(L)$  is an isomorphism in  $\mathcal{L}$ , because  $\ker(\text{ad}_L) = z(L)$  is a solvable ideal of  $L$ , we can conclude that  $\mathbf{ad}(L) \in \mathcal{L}$  where  $\mathbf{ad}$  is the covariant functor  $\mathbf{ad} : \mathcal{L} \rightarrow \mathcal{L}$  given by

$$\mathbf{ad}(L) := \text{ad}_L(L), \text{ for all } L \in \mathcal{L},$$

$$\mathbf{ad}(f)(\mathbf{ad}_L(x)) := \mathbf{ad}_{L'}(f(x)), \quad \text{for all } f \in \text{Hom}_{\mathcal{L}}(L, L').$$

Moreover, there exists a natural isomorphism  $\mathbf{ad} : \mathbf{id} \rightarrow \mathbf{ad}$ , where  $\mathbf{id} : \mathcal{L} \rightarrow \mathcal{L}$  is the identity functor and  $\mathbf{ad} := \{\mathbf{ad}_L : L \rightarrow \mathbf{ad}(L) \mid L \in \mathcal{L}\}$ . Therefore the functor  $\mathbf{ad}$  is an equivalence of categories.

Let  $\mathbf{ad}\mathcal{L}$  be the full subcategory of  $\mathcal{L}$  with objects  $\{\mathbf{ad}(L) \mid L \in \mathcal{L}\}$ .

**Lemma 3.2.** *Let  $H$  be a Lie subalgebra of  $L$ . Then,*

$$(i) \quad \mathbf{ad}_L^n(h) \Big|_H = \mathbf{ad}_H^n(h), \text{ for all } h \in H \text{ and } n \in \mathbf{Z}_+.$$

$$(ii) \quad [\mathbf{ad}_L^n(h), \mathbf{ad}_L^m(h')] \Big|_H = [\mathbf{ad}_H^n(h), \mathbf{ad}_H^m(h')], \text{ for all } h, h' \in H \text{ and } n, m \in \mathbf{Z}_+.$$

**Lemma 3.3.** *Assume that  $f \in \text{Hom}_{\mathbf{Lie}}(L, L')$  and write  $H = \text{Im}(f)$ . Then*

$$(i) \quad \mathbf{ad}_H^n(f(x))(f(y)) = f(\mathbf{ad}_L^n(x)(y)), \text{ for all } n \in \mathbf{Z}_+ \text{ and } x, y \in L.$$

$$(ii) \quad [\mathbf{ad}_H^n(f(x)), \mathbf{ad}_H^m(f(y))](f(z)) = f([\mathbf{ad}_L^n(x), \mathbf{ad}_L^m(y)](z)), \text{ for all } n, m \in \mathbf{Z}_+ \text{ and } x, y, z \in L.$$

Let  $L, L'$  be simple Lie algebras, for any non zero homomorphism  $f \in \text{Hom}_{\mathbf{Lie}^s}(L, L')$ , we will denote by  $\mathbf{ad}(f)_p$  the linear map given by

$$\begin{aligned} \mathbf{ad}(f)_p : \quad \mathbf{ad}(L)_p &\longrightarrow \mathbf{ad}(L')_p \\ \mathbf{ad}_L^{p^k}(x) &\longmapsto \mathbf{ad}_{L'}^{p^k}(f(x)) \end{aligned} .$$

From Proposition 2.10 we get  $\mathbf{ad}(L)_p$  and  $\mathbf{ad}(L')_p$  are object in  $(\mathbf{Lie}^{[p]})^s$ . Therefore, make sense to ask if  $\mathbf{ad}(f)_p$  is a  $p$ -homomorphism. To this end, let us consider a finite basis  $B = \{x_i\}_{i=1}^n$  of  $L$ . Since  $f$  is a Lie monomorphism we can conclude that  $f(B)$  is a basis of  $H = \text{Im}(f)$  and  $\mathbf{ad}(L) \cong \mathbf{ad}(H)$  in  $\mathbf{Lie}$ . Lemma 2.5 implies that  $C = \{\mathbf{ad}_L^{p^{k_i}}(x_i)\}_{i=1}^n$  and  $\mathbf{ad}(f)_p(C) = \{\mathbf{ad}_H^{p^{k_i}}(f(x_i))\}_{i=1}^n$  are a basis of  $\mathbf{ad}(L)_p$  and  $\mathbf{ad}(H)_p$  respectively, where  $\mathbf{ad}_L^{p^{k_i}}(x_i) \in C$  for all  $x_i \in B$  and some  $k_i \in \mathbf{Z}_+$ . Consequently  $\{\mathbf{ad}_{L'}^{p^{k_i}}(f(x_i))\}_{i=1}^n$  is a basis for  $\mathbf{ad}_{L'}(H)_p$ , because  $\mathbf{ad}(H) \cong \text{Im}(\mathbf{ad}(i_H)) = \mathbf{ad}_{L'}(H)$  in  $\mathbf{Lie}$ , where  $i_H : H \hookrightarrow L'$  is the canonical inclusion. Using such basis we will prove the following proposition.

**Proposition 3.4.**  *$\mathbf{ad}(f)_p$  is a  $p$ -homomorphism for any  $f \in \text{Hom}_{\mathbf{Lie}^s}(L, L')$ .*



**Proof.** To prove that  $\mathbf{ad}(f)_p$  is a morphism in **Lie**, let us consider

$$\left[ \mathbf{ad}_L^{p^{k_t}}(x_t), \mathbf{ad}_L^{p^{k_j}}(x_j) \right] = \sum_{i=1}^n \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i), \quad \alpha_i \in \mathbf{F}$$

Since  $\left[ \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)), \mathbf{ad}_{L'}^{p^{k_j}}(f(x_j)) \right] \in \mathbf{ad}_{L'}(H)_p$ ,

$$\left[ \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)), \mathbf{ad}_{L'}^{p^{k_j}}(f(x_j)) \right] = \sum_{i=1}^n \beta_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)), \quad \beta_i \in \mathbf{F}.$$

We shall prove  $\alpha_i = \beta_i$  for all  $i = 1, 2, \dots, n$ . From Lemma 3.2 it follows that

$$\begin{aligned} \left[ \mathbf{ad}_H^{p^{k_t}}(f(x_t)), \mathbf{ad}_H^{p^{k_j}}(f(x_j)) \right] &= \left[ \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)), \mathbf{ad}_{L'}^{p^{k_j}}(f(x_j)) \right] \Big|_H \\ &= \sum_{i=1}^n \beta_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)) \Big|_H \\ &= \sum_{i=1}^n \beta_i \mathbf{ad}_H^{p^{k_i}}(f(x_i)) \end{aligned}$$

On the other hand, from Lemma 3.3 we have

$$\begin{aligned} \left[ \mathbf{ad}_H^{p^{k_t}}(f(x_t)), \mathbf{ad}_H^{p^{k_j}}(f(x_j)) \right] (f(y)) &= f \left( \left[ \mathbf{ad}_L^{p^{k_t}}(x_t), \mathbf{ad}_L^{p^{k_j}}(x_j) \right] (y) \right) \\ &= f \left( \sum_{i=1}^n \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i)(y) \right) \\ &= \sum_{i=1}^n \alpha_i f \left( \mathbf{ad}_L^{p^{k_i}}(x_i)(y) \right) \\ &= \sum_{i=1}^n \alpha_i \mathbf{ad}_H^{p^{k_i}}(f(x_i))(f(y)). \end{aligned}$$

Therefore

$$\left[ \mathbf{ad}_H^{p^{k_t}}(f(x_t)), \mathbf{ad}_H^{p^{k_j}}(f(x_j)) \right] = \sum_{i=1}^n \alpha_i \mathbf{ad}_H^{p^{k_i}}(f(x_i))$$

Consequently  $\alpha_i = \beta_i$  for all  $i = 1, 2, \dots, n$  and

$$\begin{aligned} \mathbf{ad}(f)_p \left( \left[ \mathbf{ad}_L^{p^{k_t}}(x_t), \mathbf{ad}_L^{p^{k_j}}(x_j) \right] \right) &= \mathbf{ad}(f)_p \left( \sum_{i=1}^n \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) \right) \\ &= \sum_{i=1}^n \alpha_i \mathbf{ad}(f)_p \left( \mathbf{ad}_L^{p^{k_i}}(x_i) \right) \\ &= \sum_{i=1}^n \alpha_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)) \\ &= \left[ \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)), \mathbf{ad}_{L'}^{p^{k_j}}(f(x_j)) \right] \\ &= \left[ \mathbf{ad}(f)_p \left( \mathbf{ad}_L^{p^{k_t}}(x_t) \right), \mathbf{ad}(f)_p \left( \mathbf{ad}_L^{p^{k_j}}(x_j) \right) \right]. \end{aligned}$$

We proceed to show that  $\mathbf{ad}(f)_p$  is a  $p$ -homomorphism. Let us consider

$$x = \sum_{i=1}^n \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i), \quad \alpha_i \in \mathbf{F}$$

be a nonzero element in  $\mathbf{ad}(L)_p$ . Since

$$\begin{aligned} \mathbf{ad}(f)_p \left( \left( \alpha \mathbf{ad}_L^{p^k}(y) \right)^p \right) &= \mathbf{ad}(f)_p \left( \alpha^p \mathbf{ad}_L^{p^{k+1}}(y) \right) = \alpha^p \mathbf{ad}_{L'}^{p^{k+1}}(f(y)) \\ &= \left( \alpha \mathbf{ad}_{L'}^{p^k}(f(y)) \right)^p \end{aligned}$$

for all  $\alpha \in \mathbf{F}$ ,  $y \in L$  and  $k \in \mathbf{Z}_+$ . Now suppose that for  $1 < t < n$  holds

$$\mathbf{ad}(f)_p \left( \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) \right)^p \right) = \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)) \right)^p.$$

Then

$$\begin{aligned} \mathbf{ad}(f)_p \left( \left( \sum_{i=1}^t \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) \right)^p \right) &= \mathbf{ad}(f)_p \left( \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) + \alpha_t \mathbf{ad}_L^{p^{k_t}}(x_t) \right)^p \right) \\ &= \mathbf{ad}(f)_p \left( \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) \right)^p \right) \\ &\quad + \mathbf{ad}(f)_p \left( \left( \alpha_t \mathbf{ad}_L^{p^{k_t}}(x_t) \right)^p \right) + \mathbf{ad}(f)_p \\ &\quad \left( \sum_{j=1}^{p-1} s_j \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i), \alpha_t \mathbf{ad}_L^{p^{k_t}}(x_t) \right) \right) \\ &= \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)) \right)^p + \left( \alpha_t \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)) \right)^p \\ &\quad + \sum_{j=1}^{p-1} s_j \left( \sum_{i=1}^{t-1} \alpha_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)), \alpha_t \mathbf{ad}_{L'}^{p^{k_t}}(f(x_t)) \right) \\ &= \left( \sum_{i=1}^t \alpha_i \mathbf{ad}_{L'}^{p^{k_i}}(f(x_i)) \right)^p \\ &= \left( \mathbf{ad}(f)_p \left( \sum_{i=1}^t \alpha_i \mathbf{ad}_L^{p^{k_i}}(x_i) \right) \right)^p. \end{aligned}$$

The third equality follows from [17, Exercise 4, p. 69]. This completes the proof.  $\square$

**Corollary 3.5.** *There exists a covariant functor  $(-)_p : \mathbf{adLie}^s \longrightarrow (\mathbf{Lie}^{[p]})^s$  given by*

$$\begin{aligned} (-)_p(\mathbf{ad}(L)) &:= \mathbf{ad}(L)_p, \quad \text{for all } L \in \mathbf{Lie}^s, \\ (-)_p(\mathbf{ad}(f)) &:= \mathbf{ad}(f)_p, \quad \text{for all } f \in \mathbf{Hom}_{\mathbf{Lie}^s}(L, L'). \end{aligned}$$

To prove our main theorem, we will need of the following technical lemma.

**Lemma 3.6.** *If  $L \in (\mathbf{Lie}^{[p]})^s$ , then  $\mathbf{ad}(L) \in (\mathbf{Lie}^{[p]})^s$ . Moreover,  $\mathbf{ad}_L([L, L])_p = \mathbf{ad}(L)$ .*

**Proof.** From the fact that  $\text{ad}_L : L \rightarrow \mathbf{ad}(L)$  is a  $p$ -isomorphism it follows that  $\mathbf{ad}(L) \in (\mathbf{Lie}^{[p]})^s$ . Now, since for all  $\text{ad}_L(x) \in \mathbf{ad}(L)$  and  $\text{ad}_L^{p^k}([e_i, e_j]) \in \text{ad}_L([L, L])_p$

$$\begin{aligned} [\text{ad}_L(x), \text{ad}_L^{p^k}([e_i, e_j])] &= [\text{ad}_L(x), \text{ad}_L([e_i, e_j]^{[p]^k})] \\ &= \text{ad}_L([x, [e_i, e_j]^{[p]^k}]) \in \text{ad}_L([L, L]) \subset \text{ad}_L([L, L])_p \end{aligned}$$

we can conclude that  $\text{ad}_L([L, L])_p$  is an ideal of  $\mathbf{ad}(L)$  and from Proposition 2.5 we have  $\text{ad}_L([L, L])_p$  is a  $p$ -ideal of  $\mathbf{ad}(L)$ . This proves our Lemma.  $\square$

**Theorem 3.7.** *The functors  $\mathcal{F} : (\mathbf{Lie}^{[p]})^s \rightarrow \mathbf{Lie}^s$  and  $(-)_p \circ \mathbf{ad} : \mathbf{Lie}^s \rightarrow (\mathbf{Lie}^{[p]})^s$  are a pair of adjoint equivalences of categories.*

**Proof.** Denote by  $\mathcal{G}$  the functor  $(-)_p \circ \mathbf{ad}$ . Note that

$$\mathcal{F}\mathcal{G}(L) = [\mathbf{ad}(L)_p, \mathbf{ad}(L)_p] = [\mathbf{ad}(L), \mathbf{ad}(L)] = \mathbf{ad}(L), \text{ for all } L \in \mathbf{Lie}^s.$$

On the other hand, for all  $L \in (\mathbf{Lie}^{[p]})^s$  we have  $\psi_L : \mathbf{ad}([L, L])_p \rightarrow \text{ad}_L([L, L])_p$  given by  $\psi_L(\text{ad}_{[L, L]}^{p^k}(x)) = \text{ad}_L^{p^k}(x)$  is a  $p$ -isomorphism. Similarly to Proposition 3.4, it is easy to check that  $\psi_L$  is a  $p$ -homomorphism. From Lemma 3.6

$$\mathcal{G}\mathcal{F}(L) = \mathbf{ad}([L, L])_p \cong \text{ad}_L([L, L])_p = \mathbf{ad}(L), \text{ for all } L \in (\mathbf{Lie}^{[p]})^s.$$

Let us consider the following morphisms sequences in  $\mathbf{Lie}^s$  and  $(\mathbf{Lie}^{[p]})^s$  respectively

$$\eta := \{\eta_L : L \rightarrow \mathcal{F}\mathcal{G}(L)\}_{L \in \mathbf{Lie}^s} \text{ and } \nu := \{\nu_L : L \rightarrow \mathcal{G}\mathcal{F}(L)\}_{L \in (\mathbf{Lie}^{[p]})^s}$$

given by  $\eta_L := \text{ad}_L$  and  $\nu_L := \psi_L^{-1} \text{ad}_L$

$$L \xrightarrow{\text{ad}_L} \mathbf{ad}_L = \text{ad}_L([L, L])_p \xrightarrow{\psi_L^{-1}} \mathbf{ad}([L, L])_p.$$

Since  $L$  is simple (resp.  $p$ -simple) Lie algebra, we deduce that  $\eta_L$  (resp.  $\nu_L$ ) is an isomorphism (resp.  $p$ -isomorphism)

For all  $f \in \text{Hom}_{\mathbf{Lie}^s}(L, L')$  the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\eta_L} & \text{ad}(L) \\ f \downarrow & & \downarrow \mathcal{FG}(f) \\ L' & \xrightarrow{\eta_{L'}} & \text{ad}(L') \end{array}$$

because,

$$\begin{aligned} \mathcal{FG}(f) &= \mathcal{F}(\text{ad}(f)_p) = \text{ad}(f)_p \Big|_{[\text{ad}(L)_p, \text{ad}(L)_p]} = \text{ad}(f)_p \Big|_{[\text{ad}(L), \text{ad}(L)]} \\ &= \text{ad}(f)_p \Big|_{\text{ad}(L)} = \text{ad}(f). \end{aligned}$$

On the other hand, for every  $g \in \text{Hom}_{(\mathbf{Lie}^{[p]})^s}(L, L')$  the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\nu_L} & \text{ad}([L, L])_p \\ g \downarrow & & \downarrow \mathcal{GF}(g) \\ L' & \xrightarrow{\nu_{L'}} & \text{ad}([L', L'])_p \end{array}$$

because, for any  $x \in L$  (from Corollary 2.8 without loss of generality we can assume  $x = y^{[p]^k}$  with  $y \in [L, L]$  and  $k \in \mathbf{Z}_+$ )

$$\begin{aligned}
 \mathcal{GF}(g)\nu_L(x) &= \mathcal{GF}(g) \left( \psi_L^{-1} \text{ad}_L \left( y^{[p]^k} \right) \right) \\
 &= \mathcal{GF}(g) \left( \psi_L^{-1} \text{ad}_L^{p^k} (y) \right) \\
 &= \mathcal{GF}(g) \left( \text{ad}_{[L,L]}^{p^k} (y) \right) \\
 &= \mathcal{G} \left( g \Big|_{[L,L]} \right) \left( \text{ad}_{[L,L]}^{p^k} (y) \right) \\
 &= \mathbf{ad} \left( g \Big|_{[L,L]} \right)_p \left( \text{ad}_{[L,L]}^{p^k} (y) \right) \\
 &= \text{ad}_{[L',L']}^{p^k} (g(y)) \\
 &= \psi_{L'}^{-1} \text{ad}_{L'}^{p^k} (g(y)) \\
 &= \psi_{L'}^{-1} \text{ad}_{L'} \left( g \left( y^{[p]^k} \right) \right) \\
 &= \psi_{L'}^{-1} \text{ad}_{L'} (g(x)) \\
 &= \nu_{L'} g(x).
 \end{aligned}$$

Therefore, both  $\eta$  and  $\nu$  are natural isomorphism, which implies that the functors  $\mathcal{F}$  and  $\mathcal{G}$  are adjoint equivalence of categories. This proves our theorem.  $\square$

Let us mention two examples.

**Example 3.8.** For every  $L \in (\mathbf{Lie}^s)^{[p]} = \mathbf{Lie}^s \cap (\mathbf{Lie}^{[p]})^s$  we have  $\mathcal{G}(L) = \mathbf{ad}(L)$  and  $\mathcal{G}(f) = \mathbf{ad}(f)$  for all  $f \in \text{Hom}_{\mathbf{Lie}^s}(L, L')$ .

**Example 3.9.** Let us consider the simple Lie algebra  $\mathfrak{so}(3)$  with  $\text{char}(\mathbf{F}) = 2$  and the Lie homomorphism

$$\begin{aligned}
 f : \mathfrak{so}(3) &\longrightarrow \mathfrak{so}(3) \\
 e_i &\longmapsto f(e_i) := e_{4-i}, \text{ for } i = 1, 2, 3
 \end{aligned}$$

Since  $\mathfrak{so}(3) \subset \text{Der}(\mathfrak{so}(3))$  and  $\mathfrak{so}(3)_2 = \text{Der}(\mathfrak{so}(3))$  by abuse of notation

$$\begin{aligned}
 \mathbf{ad}(f)_2 : \text{Der}(\mathfrak{so}(3)) &\longrightarrow \text{Der}(\mathfrak{so}(3)) \\
 h_i &\longmapsto h_{3-i}, \text{ for } i = 1, 2 \\
 e_i &\longmapsto e_{4-i}, \text{ for } i = 1, 2, 3
 \end{aligned}$$

Note that  $e_1^{[2]} = h_1$ ,  $e_2^{[2]} = h_1 + h_2$  and  $e_3^{[2]} = h_2$ , so  $\{e_1, e_2, e_3, e_1^{[2]}, e_3^{[2]}\}$  is a basis to  $\text{Der}(\mathfrak{so}(3))$ .

It is well known that for an algebraically closed field  $\mathbf{F}$  with  $\text{char}(\mathbf{F}) = p \geq 5$ , all objects (up to isomorphism) in  $\mathbf{Lie}^s$  are classical Lie algebras, Lie algebras of generalized Cartan type, or Melikian algebras (see [10, p.

202]). As a consequence of Theorem 3.7 for an algebraically closed field with  $\text{char}(\mathbf{F}) = p \geq 5$  (up to isomorphism) all objects in  $(\mathbf{Lie}^{[p]})^s$  are minimal  $p$ -envelopes of classical Lie algebras, Lie algebras of generalized Cartan type, or Melikian algebras (using the image over the functor  $\mathcal{G} = (-)_p \circ \mathbf{ad} : \mathbf{Lie}^s \rightarrow (\mathbf{Lie}^{[p]})^s$ ). For more details on the computations of such minimal  $p$ -envelope see Example 3.8 for classical Lie algebras and [16, p. 368] for the other ones.

In contrast with  $p \geq 5$ , the classification problem of simple Lie algebras is open for  $p = 2, 3$  and one of the approach is fixing the absolute toral rank. Since the absolute toral rank of a simple Lie algebra  $L$  and the relative toral rank of  $\mathcal{G}(L)$  coincide, from Theorem 3.7 such classification problem fixing the absolute toral rank is equivalent to following problem:

**Problem 2:** *Classify all finite-dimensional  $p$ -simple Lie  $p$ -algebras of fixed relative toral rank over an algebraically closed field of characteristics 2 and 3*

Note that **Problem 1** (Section 1) and **Problem 2** are equivalent whenever the toral rank is 2.

## Acknowledgements

This research was partially supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001. The first author thanks the Institute of Mathematics and Statistics of University of Sao Paulo (IME – USP) for the hospitality and excellent working conditions, where part of this work has been done. The authors would like to give special thanks to the referee for his/her helpful comments and suggestions for to improve this paper.

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