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Results in Mathematics

Genericity of Continuous Maps with Positive Metric Mean Dimension

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Abstract. M. Gromov introduced the mean dimension for a continuous map in the late 1990's, which is an invariant under topological conjugacy. On the other hand, the notion of metric mean dimension for a dynamical system was introduced by Lindenstrauss and Weiss in 2000 and this refines the topological entropy for dynamical systems with infinite topological entropy. In this paper we will show if N is an n dimensional compact riemannian manifold then, for any $a \in [0, n]$, the set consisting of continuous maps with metric mean dimension equal to a is dense in $C^0(N)$ and for $a = n$ this set is residual. Furthermore, we prove some results related to the existence and, density of continuous maps, defined on Cantor sets, with positive metric mean dimension and also continous maps, defined on product spaces, with positive mean dimension.

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1. Introduction

Let X be a compact metric space with metric d. The notion of mean dimension for a topological dynamical system (X, ϕ), which will be denoted by $\text{mdim}(X, \phi)$, was introduced by M. Gromov in [9]. It is another invariant under topological conjugacy. Applications and properties of the mean dimension can be found in [10,11,14–17].

Lindestrauss and Weiss in [15], introduced the notion of metric mean dimension for any continuous map ϕ on X. This notion depends on the metric d on X (consequently it is not an invariant under topological conjugacy) and it is zero for any map with finite topological entropy (see [15,16,20]). Some well-known properties of the topological entropy are valid for both the mean dimension and the metric mean dimension. We will study the veracity of other fundamental and topological properties for the metric mean dimension.

In the next Section we will present the definitions of lower metric mean dimension and upper metric mean dimension of a dynamical system (X, d, ϕ), which will be denoted by $\underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi)$ and $\overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi)$, respectively. The definition of the mean dimension $\mathrm{mdim}(X, \phi)$ can be found in [15].

In Section 3 we will show the following properties of the metric mean dimension:

- It is well-known the metric mean dimension is not an invariant under topological conjugacy. In Remark 3.2 we will present an example of a path of topologically conjugate continuous maps with different metric mean dimension.
- Misiurewicz in [18] proved if ϕ has an s-horseshoe with $s \geq 2$, then $h_{\mathrm{top}}(\phi) \geq \log s$. In Theorem 3.3 we present a formula for the metric mean dimension related to the presence of horseshoes for a certain class of continuous maps on the interval. This formula allows us to provide an expression for the metric mean dimension of the compositions of a continuous map (see Corollary 3.4).
- If $\phi : X \to X$ and $\psi : Y \to Y$ are continuous maps (Y is a metric space with metric d') we have $h_{\mathrm{top}}(\phi \times \psi) = h_{\mathrm{top}}(\phi) + h_{\mathrm{top}}(\psi)$. This equality is not always valid for the (metric) mean dimension (see Example 3.8). For the mean dimension we have $\mathrm{mdim}(X \times Y, \phi \times \psi) \leq \mathrm{mdim}(X, \phi) + \mathrm{mdim}(Y, \psi)$ (see [15], Proposition 2.8). This inequality can be strict (see [21], Example 1.2). In Theorem 3.7 we will present lower and upper bounds for both $\overline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi)$ and $\underline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi)$.

Let N be a compact riemannian manifold with $n = \dim(N)$. Yano in [24] showed if $n \geq 2$, then set consisting of homeomophisms on N whose topological entropy is infinite is a residual subset of $\mathrm{Hom}(N)$. In [4], the authors proved if $n \geq 2$, then the set consisting of homeomorphisms with upper metric mean dimension equal to n is residual in $\mathrm{Hom}(N)$. In Section 4 we will show for any $a \in [0, n]$ the set consisting of continuous maps with lower and upper metric mean dimension equal to a is dense in $C^0(N)$ (see Theorems 4.1 and 4.5). Furthermore, the set consisting of continuous maps with upper metric mean dimension equal to n is residual (see Theorem 4.6). From these results we have the metric mean dimension map is not continuous anywhere on the set consisting of continuous maps defined on manifolds (see Corollaries 4.7 and 4.8).

In Section 5 we will show the existence of continuous maps on Cantor sets with positive metric mean dimension (see Proposition 5.1). Bobok and Zindulka in [3] shown the existence of homeomorphisms, defined on uncountable compact metrizable spaces with topological dimension equal to zero, with infinite topological entropy. We will use these techniques in order to prove there exist continuous maps on the Cantor set with positive metric mean dimension (see Proposition 5.1) and furthermore the density of these maps (see Theorem 5.3). Block, in [2], proved the topological entropy map is not continuous anywhere on the set consisting of continuous map on Cantor sets. This fact also holds for the metric mean dimension map (see Theorem 5.5). We will finish this work showing some results related to the density of continuous maps, defined on product spaces, with positive mean dimension (see Theorem 5.7).

2. Mean Dimension and Metric Mean Dimension

Let α be a finite open cover of a compact topological space X. Set

$$\operatorname{ord}(\alpha) = \sup_{x\in X} \sum_{U\in\alpha} 1_U(x) - 1 \qquad \text{and} \qquad \mathcal{D}(\alpha) = \min_{\beta\succ\alpha} \operatorname{ord}(\beta),$$

where 1_U is the indicator function and $\beta \succ \alpha$ means that β is a finite open cover of X finer than α. Recall that for a topological space X, the *topological dimension* is defined as

$$\dim(X) = \sup_{\alpha} \mathcal{D}(\alpha),$$

where α runs over all finite open covers of X. For any continuous map $\phi : X \to X$, define

$$\alpha_0^{n-1} = \alpha \vee (\phi^{-1}(\alpha)) \vee (\phi^{-2}(\alpha)) \vee \cdots \vee (\phi^{-n+1}(\alpha)).$$

Definition 2.1. The *mean dimension* of $\phi : X \to X$ is defined to be

$$\operatorname{mdim}(X, \phi) = \sup_{\alpha} \lim_{n\to\infty} \frac{\mathcal{D}(\alpha_0^{n-1})}{n},$$

where α runs over all finite open covers of X.

If $\dim(X) < \infty$, then $\operatorname{mdim}(X, \phi) = 0$ (see [15]). Furthermore, in [15], Proposition 3.1, is proved that $\operatorname{mdim}(X^{\mathbb{Z}}, \sigma) \le \dim(X)$, where σ is the shift map on $X^{\mathbb{Z}}$.

Let X be a compact metric space endowed with a metric d and $\phi : X \to X$ a continuous map. For any non-negative integer n we define $d_n : X \times X \to [0, \infty)$ by

$$d_n(x, y) = \max\{d(x, y), d(\phi(x), \phi(y)), \dots, d(\phi^{n-1}(x), \phi^{n-1}(y))\}.$$

Fix $\varepsilon > 0$. We say that $A \subset X$ is an (n, ϕ, ε)-*separated* set if $d_n(x, y) > \varepsilon$, for any two distinct points $x, y \in A$. We denote by $\operatorname{sep}(n, \phi, \varepsilon)$ the maximal cardinality of an (n, ϕ, ε)-separated subset of X. We say that $E \subset X$ is an (n, ϕ, ε)-*spanning* set for X if for any $x \in X$ there exists $y \in E$ such that $d_n(x, y) < \varepsilon$. Let $\operatorname{span}(n, \phi, \varepsilon)$ be the minimum cardinality of any (n, ϕ, ε)-spanning subset of X. Given an open cover α, we say that α is an (n, ϕ, ε)-*cover* of X if the d_n-diameter of any element of α is less than ε. Let $\operatorname{cov}(n,\phi,\varepsilon)$ be the minimum number of elements in any (n,ϕ,ε)-cover of X. Set

- $\operatorname{sep}(\phi,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{sep}(n,\phi,\varepsilon)$;
- $\operatorname{span}(\phi,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{span}(n,\phi,\varepsilon)$;
- $\operatorname{cov}(\phi,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{cov}(n,\phi,\varepsilon)$.

Definition 2.2. The *topological entropy* of (X,ϕ,d) is defined by

$$h_{\mathrm{top}}(\phi)=\lim_{\varepsilon\to 0}\operatorname{sep}(\phi,\varepsilon)=\lim_{\varepsilon\to 0}\operatorname{span}(\phi,\varepsilon)=\lim_{\varepsilon\to 0}\operatorname{cov}(\phi,\varepsilon).$$

Definition 2.3. The *lower metric mean dimension* and the *upper metric mean dimension* of (X,d,ϕ) are defined by

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X,d,\phi)=\liminf_{\varepsilon\to 0}\frac{\operatorname{sep}(\phi,\varepsilon)}{|\log\varepsilon|}=\liminf_{\varepsilon\to 0}\frac{\operatorname{span}(\phi,\varepsilon)}{|\log\varepsilon|}=\liminf_{\varepsilon\to 0}\frac{\operatorname{cov}(\phi,\varepsilon)}{|\log\varepsilon|}$$
$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X,d,\phi)=\limsup_{\varepsilon\to 0}\frac{\operatorname{sep}(\phi,\varepsilon)}{|\log\varepsilon|}=\limsup_{\varepsilon\to 0}\frac{\operatorname{span}(\phi,\varepsilon)}{|\log\varepsilon|}=\limsup_{\varepsilon\to 0}\frac{\operatorname{cov}(\phi,\varepsilon)}{|\log\varepsilon|},$$

respectively.

Remark 2.4. Throughout the paper, we will omit the underline and the overline on the notations $\underline{\mathrm{mdim}}_{\mathrm{M}}$ and $\overline{\mathrm{mdim}}_{\mathrm{M}}$ when a result be valid for both cases, that is, we will use $\mathrm{mdim}_{\mathrm{M}}$ for the both cases.

3. Some Fundamental Properties

One of the most important properties of the topological entropy is that it is an invariant under topological conjugacy. Mean dimension is an invariant under topological conjugacy (see [15]). It is well-known the metric mean dimension for continuous maps depends on the metric d on X. Consequently, it is not an invariant under topological conjugacy between dynamical systems. In the next example we will show that we can find paths of continuous maps that are topologically conjugate and have different metric mean dimension.

Example 3.1. Fix $r\in(0,\infty)$. Set $a_0=0$ and $a_n=\sum_{i=0}^{n-1}\frac{C}{3^{ir}}$ for $n\geq 1$, where $C=\frac{1}{\sum_{i=0}^{\infty}\frac{1}{3^{ir}}}=\frac{3^r-1}{3^r}$. For each $n\geq 0$, let $T_n:I_n:=[a_n,a_{n+1}]\to[0,1]$ be the unique increasing affine map from I_n onto $[0,1]$.

For $s\in\mathbb{N}$, set $\phi_{s,r}:[0,1]\to[0,1]$, given by $\phi_{s,r}|_{I_n}=T_n^{-1}\circ g^{s(n+1)}\circ T_n$ for any $n\geq 0$, where $g:[0,1]\to[0,1]$, is defined by $x\mapsto|1-|3x-1||$ (see Fig. 1). We will prove that

$$\mathrm{mdim}_{\mathrm{M}}([0,1],|\cdot|,\phi_{s,r})=\frac{s}{r+s}\quad\text{for any } s\in\mathbb{N}.$$

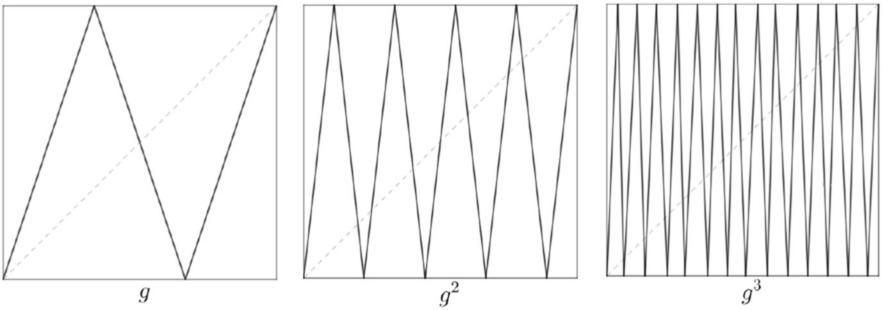


FIGURE 1. Graphs of g, g^2, g^3

Take any $\varepsilon \in (0,1)$. For any $k \geq 1$ set $\varepsilon_k = \frac{|I_k|}{3^{s(k+1)}} = \frac{C}{3^{k(r+s)+s}}$, where $|I_k| = a_{k+1} - a_k$. There exists some $k \geq 1$ such that $\varepsilon \in [\varepsilon_k, \varepsilon_{k-1}]$. Note that

$$\mathrm{sep}(n, \phi_{s,r}, \varepsilon) \geq \mathrm{sep}(n, \phi_{s,r}, \varepsilon_{k-1}) \geq \mathrm{sep}(n, \phi_{s,r}|_{I_{k-1}}, \varepsilon_{k-1}) \quad \text{for any } n \geq 1.$$

From Lemma 6 in [22] it follows that for any $n \geq 1$ we have

$$\mathrm{sep}(n, \phi_{s,r}|_{I_{k-1}}, \varepsilon_{k-1}) \geq \left(\frac{3^{sk}}{2}\right)^n \quad \text{and hence} \quad \mathrm{sep}(\phi_{s,r}, \varepsilon) \geq \log\left(\frac{3^{sk}}{2}\right).$$

Thus

$$\begin{aligned}\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s,r}) &= \liminf_{\varepsilon \to 0} \frac{\mathrm{sep}(\phi_{s,r}, \varepsilon)}{|\log \varepsilon|} \geq \lim_{k\to\infty} \frac{\log 3^{sk}}{|\log \varepsilon_k|} \\ &= \lim_{k\to\infty} \frac{\log 3^{sk}}{\log 3^{k(r+s)+s}} = \frac{s}{r+s}.\end{aligned}$$

On the other hand, note that $\frac{s(k+1)\log 3}{((k-1)(r+s)+s)\log 3 - \log C} \to \frac{s}{r+s}$ as $k \to \infty$. Hence, for any $\delta > 0$ there exists $k_0 \geq 1$ such that for any $k > k_0$ we have $\frac{s(k+1)\log 3}{((k-1)(r+s)+s)\log 3 - \log C} < \frac{s}{r+s} + \delta$. Hence, suppose that ε is small enough such that $\varepsilon < \varepsilon_{k_0 - 1}$. Let $k \geq k_0$ such that $\varepsilon \in [\varepsilon_k, \varepsilon_{k-1}]$. For each $0 \leq j \leq k$, dividing each I_j into $\frac{3^{s(j+1)n}|I_j|}{\varepsilon}$ sub-intervals with the same length, we have the set consisting of the end points of these sub-intervals is an $(n, \phi_{s,r}, \varepsilon)$-spanning set (see [5], Corollary 7.2). Hence, if $Y_k = \cup_{j=0}^k I_j$, for every $n \geq 1$ we have

$$\begin{aligned}\mathrm{span}(n, \phi_{s,r}|_{Y_k}, \varepsilon) &\leq \sum_{j=0}^{k} \frac{3^{s(j+1)n}|I_j|}{\varepsilon} \leq \sum_{j=0}^{k} \frac{3^{s(j+1)n}|I_j|}{\varepsilon_k} = \sum_{j=0}^{k} \frac{3^{s(j+1)n}3^{s(k+1)}|I_j|}{|I_k|} \\ &= \sum_{j=0}^{k} \frac{3^{sn(j+1)}3^{s(k+1)}3^{kr}}{3^{jr}} \leq (k+1)3^{s(k+1)n}3^{s(k+1)+kr}.\end{aligned}$$

Hence

$$\frac{\mathrm{span}(\phi_{s,r}|_{Y_k}, \varepsilon)}{|\log \varepsilon|} \leq \limsup_{n\to\infty} \frac{\log[(k+1)3^{s(k+1)n}3^{s(k+1)+kr}]}{n|\log \varepsilon_{k-1}|}$$

$$= \limsup_{n\to\infty} \frac{s(k+1)\log 3}{[((k-1)(r+s)+s)\log 3 - \log C]}$$
$$= \frac{s(k+1)\log 3}{((k-1)(r+s)+s)\log 3 - \log C} < \frac{s}{s+r} + \delta.$$

This fact implies that for any $\delta > 0$ we have

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s,r}) < \frac{s}{r+s} + \delta \quad \text{and hence} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s,r}) \leq \frac{s}{r+s}$$

The above facts proves $\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s,r}) = \frac{s}{r+s}$.

Note for each $s \geq 1$ and $r \in (0,1)$ we have

$$\phi_{s,r} = \phi_{1,r}^s. \tag{3.1}$$

Hence, in this case we have

$$\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{1,r}^s) = \frac{s}{r+s} = \frac{s\,\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{1,r})}{\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{1,r})(s+1) - 1} \quad \text{for each } s \in \mathbb{N}.$$

Remark 3.2. Let $r_1 > 0$ and $r_2 > 0$. For each $n \geq 1$, take $I_n^{r_1}$ and $I_n^{r_2}$ the intervals obtained as in the Example 3.1, for r_1 and r_2, respectively. Fix $s \geq 1$ and let ϕ_{s,r_1} and ϕ_{s,r_2} be the continuous maps defined above for r_1 and r_2, respectively. Note that, for each $n \geq 0$, $\phi_{s,r_1}|_{I_n^{r_1}}$ and $\phi_{s,r_2}|_{I_n^{r_2}}$ are topologically conjugate by a continuous map $h_n : I_n^{r_1} \to I_n^{r_2}$:

$$\phi_{s,r_1}|_{I_n^{r_1}} = h_n^{-1} \circ \phi_{s,r_2}|_{I_n^{r_2}} \circ h_n.$$

Therefore, ϕ_{s,r_1} and ϕ_{s,r_2} are topologically conjugate by $h : I \to I$ given by $h|_{I_n^{r_1}} = h_n$ for each $n \geq 0$. This fact proves, for each $s \in \mathbb{N}$, $\mathcal{A}_s = \{\phi_{s,r} : r \in (0,\infty)\}$ is a path of topologically conjugate continuous maps such that $\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s,r}) = \frac{s}{r+s}$ for each $r \in (0,\infty)$.

An *s-horseshoe* for $\phi : [0,1] \to [0,1]$ is an interval $J \subseteq [0,1]$ which has a partition into s subintervals $J_1, \ldots, J_s$ such that $J_j^\circ \cap J_i^\circ = \emptyset$ for $i \neq j$ and $J \subseteq \phi(\overline{J}_i)$ for each $i = 1, \ldots, s$.

If g is the map defined in Example 3.1, we have $I = [0,1]$ is an 3-horseshoe for g. Furthermore, for $n \geq 0$, each I_n can be divided into $3^{s(n+1)}$ closed intervals with the same length $I_n^1, \ldots, I_n^{3^{s(n+1)}}$, such that

$$\phi_{s,r}(I_n^i) = I_n \quad \text{for each } i \in \{1, \ldots, 3^{s(n+1)}\}.$$

Consequently, each I_n is a $3^{s(n+1)}$-horseshoe for $\phi_{s,r}$.

Misiurewicz in [18], proved if ϕ has an s-horseshoe with $s \geq 2$, then $h_{\mathrm{top}}(\phi) \geq \log s$. Suppose for each $k \in \mathbb{N}$ there exists an s_k-horseshoe for $\phi \in C^0([0,1])$, $I_k = [a_{k-1}, a_k] \subseteq [0,1]$, consisting of sub-intervals $I_k^1, I_k^2, \ldots, I_k^{s_k}$

with the same length, where $s_k \geq 2$ for all $k \geq 1$. From Lemma 6 in [22] we can prove that (see Example 3.1)

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \geq \limsup_{k\to\infty} \frac{1}{\left|1 - \frac{\log|I_k|}{\log s_k}\right|}. \tag{3.2}$$

Next theorem provides upper bounds for the lower metric mean dimension.

Theorem 3.3. *Suppose for each $k \in \mathbb{N}$ there exists a s_k-horseshoe for $\phi \in C^0([0,1])$, $I_k = [a_{k-1}, a_k] \subseteq [0,1]$, consisting of sub-intervals with the same length $I_k^1, I_k^2, \ldots, I_k^{s_k}$ and $[0,1] = \cup_{k=1}^{\infty} I_k$. We can rearrange the intervals and suppose that $2 \leq s_k \leq s_{k+1}$ for each k. If each $\phi|_{I_k^i} : I_k^i \to I_k$ is a bijective affine map for all k and $i = 1, \ldots, s_k$, we have*

i. $\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \leq \liminf_{k\to\infty} \frac{1}{\left|1-\frac{\log|I_k|}{\log s_k}\right|}$.

ii. If the limit $\lim_{k\to\infty} \frac{1}{\left|1-\frac{\log|I_k|}{\log s_k}\right|}$ *exists, then* $\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = \lim_{k\to\infty} \frac{1}{\left|1-\frac{\log|I_k|}{\log s_k}\right|}$.

Proof. Let k_i be a strictly increasing sequence of positive integers such that

$$a := \liminf_{k\to\infty} \frac{1}{\left|1 - \frac{\log|I_k|}{\log s_k}\right|} = \lim_{i\to\infty} \frac{1}{\left|1 - \frac{\log|I_{k_i}|}{\log s_{k_i}}\right|}.$$

For any $\delta > 0$, there exists k_0 such that if, $k_i \geq k_0$, then $\frac{1}{\left|1-\frac{\log|I_{k_i}|}{\log s_{k_i}}\right|} < a + \delta$.

For any $k_i \geq k_0$, set $\varepsilon_{k_i} = \frac{|I_{k_i}|}{s_{k_i}}$. For each $1 \leq j \leq k_i$, dividing each I_j into $\frac{s_j^n |I_j|}{\varepsilon_{k_i}}$ sub-intervals with the same length, we have the set consisting of the end points of these sub-intervals is an $(n, \phi|_{I_j}, \varepsilon_{k_i})$-spanning set (see [5], Corollary 7.2). Hence, if $Y_{k_i} = \cup_{j=1}^{k_i} I_j$, for every $n \geq 1$ we have

$$\mathrm{span}(n, \phi|_{Y_{k_i}}, \varepsilon_{k_i}) \leq \sum_{j=1}^{k_i} \frac{s_j^n |I_j|}{\varepsilon_{k_i}} \leq \sum_{j=1}^{k_i} \frac{s_{k_i}^n |I_j|}{\varepsilon_{k_i}}.$$

Thus

$$\frac{\mathrm{span}(\phi|_{Y_{k_i}}, \varepsilon_{k_i})}{|\log \varepsilon_{k_i}|} \leq \limsup_{n\to\infty} \frac{\log\left[\sum_{j=1}^{k_i} \frac{s_{k_i}^n |I_j|}{\varepsilon_{k_i}}\right]}{n|\log s_{k_i} - \log|I_{k_i}||} = \frac{1}{\left|1 - \frac{\log|I_{k_i}|}{\log s_{k_i}}\right|} < a + \delta$$

This fact implies that for any $\delta > 0$ we have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \leq a + \delta \quad \text{and hence} \quad \underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \leq a,$$

which proves i.

Next, we will prove ii. From (3.2) we have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \leq \lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log|I_k|}{\log s_k}\right|} \leq \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi). \tag{3.3}$$

We can prove that for any $\delta > 0$ there exists k_0 such that for any $k \geq k_0$ we have

$$\frac{\operatorname{span}(\phi|_{Y_k}, \varepsilon)}{|\log \varepsilon|} \leq \frac{\operatorname{span}(\phi|_{Y_k}, \varepsilon)}{|\log \varepsilon_k|} \leq \frac{\log s_k}{|\log s_k - \log |I_k||} = \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|} < a + \delta,$$

for any $\varepsilon > 0$ small enough such that $\varepsilon \leq \varepsilon_k$ (see Example 3.1). Hence

$$\overline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \leq \lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}.$$

The equality follows from (3.3). $\square$

It is well-known that for any continuous map $\phi : X \to X$ and $s \in \mathbb{N}$ we have

$$\operatorname{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi^s) \leq s \operatorname{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi)$$

and this inequality can be strict. Next corollary, which follows directly from Theorem 3.3, provides a formula for the metric mean dimension of the compositions of a map satisfying the conditions of the theorem.

Corollary 3.4. *If ϕ is a map which satisfies the properties of Theorem 3.3 then for any $s \in \mathbb{N}$ we have*

$$\overline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi^s) \geq \limsup_{k\to\infty} \frac{s}{\left|s - \frac{\log |I_k|}{\log s_k}\right|} \quad \textit{and} \quad \underline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi^s) \leq \liminf_{k\to\infty} \frac{s}{\left|s - \frac{\log |I_k|}{\log s_k}\right|}.$$

If the limit $\lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}$ *exists, then* $\overline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi^s) = \lim_{k\to\infty} \frac{s}{\left|s - \frac{\log |I_k|}{\log s_k}\right|}$.

Note that in Example 3.1, for each map $\phi_{s,r}$ we have

$$\lim_{k\to\infty} \frac{\log |I_k|}{\log s_k} = -\lim_{k\to\infty} \frac{\log 3^{kr}}{\log 3^{s(k+1)}} = -\frac{r}{s}.$$

Example 3.5. Set $a_0 = 0$ and $a_n = \sum_{i=1}^{n} \frac{6}{\pi^2 i^2}$ for $n \geq 1$. Set $I_n := [a_{n-1}, a_n]$ for any $n \geq 1$. Let $\varphi \in C^0([0,1])$ be defined by $\varphi|_{I_n} = T_n^{-1} \circ g^n \circ T_n$ for any $n \geq 1$, where T_n and g are as in Example 3.1 (see Example 3.4 in [20]). For φ^s, with $s \in \mathbb{N}$, we have $s_k = 3^{sk}$ for each $k \in \mathbb{N}$. Therefore,

$$\lim_{k\to\infty} \frac{\log |I_k|}{\log s_k} = -\lim_{k\to\infty} \frac{\log k^2}{\log 3^{sk}} = 0 \quad \text{for any } s \in \mathbb{N}.$$

It is follows from Theorem 3.3 that

$$\overline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi^s) = \lim_{k\to\infty} \frac{s}{\left|s - \frac{\log |I_k|}{\log s_k}\right|} = 1 \quad \text{for any } s \in \mathbb{N}.$$

The equality $\underline{\operatorname{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi^s) = 1$ can be proved as in Example 3.1.

Example 3.6. Take $I_n = [a_{n-1}, a_n]$ as in the above example. Divide each interval I_n into $2n+1$ sub-intervals with the same lenght, $I_n^1, \ldots, I_n^{2n+1}$. For $k = 1, 3, \ldots, 2n+1$, let $\psi|_{I_n^k} : I_n^k \to I_n$ be the unique increasing affine map from I_n^k onto I_n and for $k = 2, 4, \ldots, 2n$, let $\psi|_{I_n^k} : I_n^k \to I_n$ be the unique decreasing affine map from I_n^k onto I_n. For ψ^s, with $s \in \mathbb{N}$, we have $|I_k| = \frac{6}{\pi^2 k^2}$ and $s_k = (2k+1)^s$ for each $k \in \mathbb{N}$. Therefore,

$$\lim_{k\to\infty} \frac{\log |I_k|}{\log s_k} = -\lim_{k\to\infty} \frac{\log k^2}{\log(2k+1)^s} = -\frac{2}{s}.$$

It follows from Theorem 3.3 that

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \psi^s) = \lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|} = \frac{s}{s+2} \quad \text{for any } s \in \mathbb{N}.$$

The equality $\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \psi^s) = \frac{s}{s+2}$ can be proved as in Example 3.1.

Take $\phi : X \to X$ and $\psi : Y \to Y$ where Y is a compact metric space with metric d'. On $X \times Y$ we consider the metric

$$(d \times d')((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2), \quad \text{for } x_1, x_2 \in X \text{ and } y_1, y_2 \in Y. \tag{3.4}$$

The map $\phi \times \psi : X \times Y \to X \times Y$ is defined to be $(\phi \times \psi)(x, y) = (\phi(x), \psi(y))$ for any $(x, y) \in X \times Y$. The equality $h_{\mathrm{top}}(\phi \times \psi) = h_{\mathrm{top}}(\phi) + h_{\mathrm{top}}(\psi)$ always hold. Lindenstrauss in [15], Proposition 2.8, proved that

$$\mathrm{mdim}(X \times Y, \phi \times \psi) \leq \mathrm{mdim}(X, \phi) + \mathrm{mdim}(Y, \psi) \tag{3.5}$$

and this inequality can be strict (see [13,21]). For the metric mean dimension we have:

Theorem 3.7. *Take two continuous maps $\phi : X \to X$ and $\psi : Y \to Y$. On $X \times Y$ consider the metric given in (3.4). We have:*

i. $\overline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi) \leq \overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \overline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi)$.
ii. $\overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \underline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi) \leq \overline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi)$.
iii. $\underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \underline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi)$.
iv. $\underline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \overline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi)$.
v. *If* $\overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) = \underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi)$ *or* $\overline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi) = \underline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi)$, *then*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi) = \overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \overline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi)$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X \times Y, d \times d', \phi \times \psi) = \underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) + \underline{\mathrm{mdim}}_{\mathrm{M}}(Y, d', \psi).$$

Proof. For any $\varepsilon > 0$, we always have

$$\mathrm{span}(\phi \times \psi, 2\varepsilon) \leq \mathrm{span}(\phi, \varepsilon) + \mathrm{span}(\psi, \varepsilon) \quad \text{and} \quad \mathrm{sep}(\phi \times \psi, 2\varepsilon) \geq \mathrm{sep}(\phi, \varepsilon) + \mathrm{sep}(\psi, \varepsilon).$$

Hence, item i. follows from

$$\limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\phi\times\psi,2\varepsilon)}{|\log 2\varepsilon|}\leq \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\phi,\varepsilon)}{|\log\varepsilon|}+\limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\psi,\varepsilon)}{|\log\varepsilon|},$$

item ii. follows from

$$\limsup_{\varepsilon\to 0}\frac{\mathrm{sep}(\phi\times\psi,2\varepsilon)}{|\log 2\varepsilon|}$$
$$\geq \limsup_{\varepsilon\to 0}\frac{\mathrm{sep}(\phi,\varepsilon)}{|\log\varepsilon|}+\liminf_{\varepsilon\to 0}\frac{\mathrm{sep}(\psi,\varepsilon)}{|\log\varepsilon|},$$

item iii. follows from

$$\liminf_{\varepsilon\to 0}\frac{\mathrm{sep}(\phi\times\psi,2\varepsilon)}{|\log 2\varepsilon|}\geq \liminf_{\varepsilon\to 0}\frac{\mathrm{sep}(\phi,\varepsilon)}{|\log\varepsilon|}+\liminf_{\varepsilon\to 0}\frac{\mathrm{sep}(\psi,\varepsilon)}{|\log\varepsilon|}$$

and item iv. follows from

$$\liminf_{\varepsilon\to 0}\frac{\mathrm{span}(\phi\times\psi,2\varepsilon)}{|\log 2\varepsilon|}\leq \liminf_{\varepsilon\to 0}\frac{\mathrm{span}(\phi,\varepsilon)}{|\log\varepsilon|}+\limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\psi,\varepsilon)}{|\log\varepsilon|}.$$

Note that item v. is a consequence of items i-iv. $\square$

For the box dimension we have the following inequalities (see [8,23]):

$$\overline{\dim}_{\mathrm{B}}(X,d)+\underline{\dim}_{\mathrm{B}}(Y,d')\leq\overline{\dim}_{\mathrm{B}}(X\times Y,d\times d')\leq\overline{\dim}_{\mathrm{B}}(X,d)+\overline{\dim}_{\mathrm{B}}(Y,d') \tag{3.6}$$

and

$$\underline{\dim}_{\mathrm{B}}(X,d)+\underline{\dim}_{\mathrm{B}}(Y,d')\leq\underline{\dim}_{\mathrm{B}}(X\times Y,d\times d')\leq\underline{\dim}_{\mathrm{B}}(X,d)+\overline{\dim}_{\mathrm{B}}(Y,d'). \tag{3.7}$$

If $\overline{\dim}_{\mathrm{B}}(X,d)=\underline{\dim}_{\mathrm{B}}(X,d)$ or $\overline{\dim}_{\mathrm{B}}(Y,d')=\underline{\dim}_{\mathrm{B}}(Y,d')$, we can prove that

$$\overline{\dim}_{\mathrm{B}}(X\times Y,d\times d')=\overline{\dim}_{\mathrm{B}}(X,d)+\overline{\dim}_{\mathrm{B}}(Y,d')$$

and

$$\underline{\dim}_{\mathrm{B}}(X\times Y,d\times d')=\underline{\dim}_{\mathrm{B}}(X,d)+\underline{\dim}_{\mathrm{B}}(Y,d').$$

Each inequality in (3.6) and (3.7) can be strict (see [23]). In the next example we will prove the inequalities i-iv in Theorem 3.7 can be strict.

Example 3.8. Let (X,d) and (Y,d') be any compact metric spaces. The metric $\tilde{d}\times\tilde{d}'$ on $X^{\mathbb{Z}}\times Y^{\mathbb{Z}}$ is defined by

$$(\tilde{d}\times\tilde{d}')((\bar{x},\bar{y}),(\bar{z},\bar{w}))=\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}d(x_i,z_i)+\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}d'(y_i,w_i),$$

for $\bar{x}=(x_i)_{i\in\mathbb{Z}}$, $\bar{z}=(z_i)_{i\in\mathbb{Z}}\in X^{\mathbb{Z}}, \bar{y}=(y_i)_{i\in\mathbb{Z}}$, $\bar{w}=(w_i)_{i\in\mathbb{Z}}\in Y^{\mathbb{Z}}$. Furthermore, the metric $(d\times d')^*$ on $(X\times Y)^{\mathbb{Z}}$ is given by

$$(d\times d')^*((\overline{x,y}),(\overline{z,w}))=\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}d(x_i,z_i)+\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}d'(y_i,w_i),$$

for $(\overline{x,y}) = (x_i, y_i)_{i\in\mathbb{Z}}$ and $(\overline{z,w}) = (z_i, w_i)_{i\in\mathbb{Z}}$ in $(X \times Y)^{\mathbb{Z}}$. Consequently, the bijection

$$\Theta : (X \times Y)^{\mathbb{Z}} \to X^{\mathbb{Z}} \times Y^{\mathbb{Z}}, \quad \text{given by } (x_i, y_i)_{i\in\mathbb{Z}} \mapsto ((x_i)_{i\in\mathbb{Z}}, (y_i)_{i\in\mathbb{Z}}),$$

is an isometry and furthermore the diagram

$$\begin{array}{ccc} (X\times Y)^{\mathbb{Z}} & \xrightarrow{\sigma} & (X\times Y)^{\mathbb{Z}} \\ \Big\downarrow \Theta & & \Big\downarrow \Theta \\ X^{\mathbb{Z}} \times Y^{\mathbb{Z}} & \xrightarrow{\sigma_1\times\sigma_2} & X^{\mathbb{Z}} \times Y^{\mathbb{Z}} \end{array}$$

is commutative, where σ is the left shift on $(X \times Y)^{\mathbb{Z}}$, σ_1 is the left shift on $X^{\mathbb{Z}}$ and σ_2 is the left shift on $Y^{\mathbb{Z}}$. It is clear that the metric mean dimension is invariant under isometric topological conjugacy. Therefore,

$$\begin{aligned} \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{Z}} \times Y^{\mathbb{Z}}, \tilde{d}_1 \times \tilde{d}_2, \sigma_1 \times \sigma_2) &= \overline{\mathrm{mdim}}_{\mathrm{M}}((X \times Y)^{\mathbb{Z}}, (d \times d')^*, \sigma) \\ &= \overline{\dim}_{\mathrm{B}}(X \times Y, d \times d') \end{aligned}$$

and

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{Z}} \times Y^{\mathbb{Z}}, \tilde{d}_1 \times \tilde{d}_2, \sigma_1 \times \sigma_2) = \underline{\dim}_{\mathrm{B}}(X \times Y, d \times d').$$

If (X, d) and (Y, d') are compact metric spaces such that each inequality in (3.6) and (3.7) is strict (see [23]), then we can prove that the inequalities i-iv in Theorem 3.7 are strict. For instances, if

$$\overline{\dim}_{\mathrm{B}}(X \times Y, d \times d') < \overline{\dim}_{\mathrm{B}}(X, d) + \overline{\dim}_{\mathrm{B}}(Y, d'),$$

then

$$\begin{aligned} \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{Z}} \times Y^{\mathbb{Z}}, \tilde{d}_1 \times \tilde{d}_2, \sigma_1 \times \sigma_2) &= \overline{\dim}_{\mathrm{B}}(X \times Y, d \times d') < \overline{\dim}_{\mathrm{B}}(X, d) + \overline{\dim}_{\mathrm{B}}(Y, d') \\ &= \overline{\mathrm{mdim}}_{\mathrm{M}}(X^{\mathbb{Z}}, \tilde{d}, \sigma_1) + \overline{\mathrm{mdim}}_{\mathrm{M}}(Y^{\mathbb{Z}}, \tilde{d}', \sigma_2). \end{aligned}$$

4. Density of Continuous Maps on Manifolds with Positive Metric Mean Dimension

Yano in [24] proved that the set consisting of homeomorphisms with infinite topological entropy defined on any manifold with dimension biggest to one is residual in the set consisting of homeomorphisms on the manifold. Furthermore, the set consisting of continuous maps defined on the interval or the circle with infinite topological entropy is residual. In this section we will prove if N is any riemannian manifold, then for any $a \in [0, \dim(N)]$ the set consisting of continuous maps on N whose metric mean dimension is equal to a is dense in $C^0(N)$. Furthermore, the set consisting of continuous maps with upper metric mean dimension equal to $\dim(N)$ is residual.

On $C^0(X)$ we will consider the metric

$$\hat{d}(\phi, \varphi) = \max_{x\in X} d(\phi(x), \varphi(x)) \qquad \text{for any } \phi, \varphi \in C^0(X).$$

For any $a \in [0, \overline{\dim}_{\mathrm{B}}(X, d)]$, set

$$C_a(X) = \{\phi \in C^0(X) : \overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) = \underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) = a\}.$$

Note for any riemannian manifold N with riemannian metric d, we have $C_0(N)$ is dense in $C^0(N)$, since the set consisting of C^1-maps on N is dense in $C^0(N)$ and the metric mean dimension of any C^1-map is equal to zero.

Examples 3.1 and 3.5 prove for any $a \in [0, 1]$ there exists a $\phi_a \in C^0([0, 1])$ such that

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0, 1], |\cdot|, \phi_a) = \underline{\mathrm{mdim}}_{\mathrm{M}}([0, 1], |\cdot|, \phi_a) = a.$$

In [4], Theorem C, the authors proved for each $a \in [0, 1]$ the set consisting of continuous maps on $[0, 1]$ with lower and upper metric mean dimension equal to a is dense in $C^0([0, 1])$. We will present a proof of this fact for the sake of completeness.

Theorem 4.1. *$C_a([0, 1])$ is dense in $C^0([0, 1])$ for each $a \in [0, 1]$.*

Proof. We had seen that $C_0([0, 1])$ is dense in $C^0([0, 1])$. Therefore, in order to prove the theorem, it is sufficient to show if $\phi_0 \in C_0([0, 1])$, then for any $\varepsilon > 0$ there exists $\psi_a \in C_a([0, 1])$ such that $d(\phi_0, \psi_a) < \varepsilon$. Fix $\phi_0 : [0, 1] \to [0, 1] \in C_0([0, 1])$ and take $\varepsilon > 0$.

Let p^* be a fixed point of ϕ_0. Choose $\delta > 0$ such that $|\phi_0(x) - \phi_0(p^*)| < \varepsilon/2$ for any x with $|x - p^*| < \delta$. Take $\phi_a \in C_a([0, 1])$ for some $a \in (0, 1]$ (it is follows from Examples 3.1 and 3.5 that for any $a \in (0, 1]$ there exists $\phi_a \in C_a([0, 1])$). Set $J_1 = [0, p^*]$, $J_2 = [p^*, p^* + \delta/2]$, $J_3 = [p^* + \delta/2, p^* + \delta]$ and $J_4 = [p^* + \delta, 1]$. Take the continuous map ψ_a on X defined as

$$\psi_a(x) = \begin{cases} \phi_0(x), & \text{if } x \in J_1 \cup J_4, \\ T_2^{-1}\phi_a T_2(x), & \text{if } x \in J_2, \\ T_3(x), & \text{if } x \in J_3, \end{cases}$$

where $T_2 : J_2 \to I$ is the affine map such that $T_2(p^*) = 0$ and $T_2(p^* + \delta/2) = 1$, and $T_3 : J_3 \to [p^* + \delta/2, \phi_0(p^* + \delta)]$ is the affine map such that $T_3(p^* + \delta/2) = p^* + \delta/2$ and $T_3(p^* + \delta) = \phi_0(p^* + \delta)$ (see Fig. 2). Note that $\hat{d}(\psi_a, \phi_0) < \varepsilon$. Set $A = \cup_{i=0}^{\infty} \psi_a^{-i}(J_2)$ and $B = A^c$. Note that

$$\Omega(\psi_a|_A) = \Omega(\psi_a|_{J_2}) \subseteq J_2.$$

Hence

$$\begin{aligned} \mathrm{mdim}_{\mathrm{M}}([0, 1], |\cdot|, \psi_a) &= \max\{\mathrm{mdim}_{\mathrm{M}}(A, |\cdot|, \psi_a|_A), \mathrm{mdim}_{\mathrm{M}}(B, |\cdot|, \psi_a|_B)\} \\ &= \mathrm{mdim}_{\mathrm{M}}(J_2, |\cdot|, \psi_a) = a. \end{aligned}$$

This fact proves the theorem. □

Remark 4.2. Note that in Theorem 4.1 we prove the set $\mathcal{A}$ consisting of maps $\psi \in C^0([0, 1])$ such that, for some $a, b \in [0, 1]$, $\psi|_{[a,b]} : [a, b] \to [a, b]$ satisfies the conditions in Theorem 3.3 and outside of $[a, b]$ ψ has zero entropy, is dense

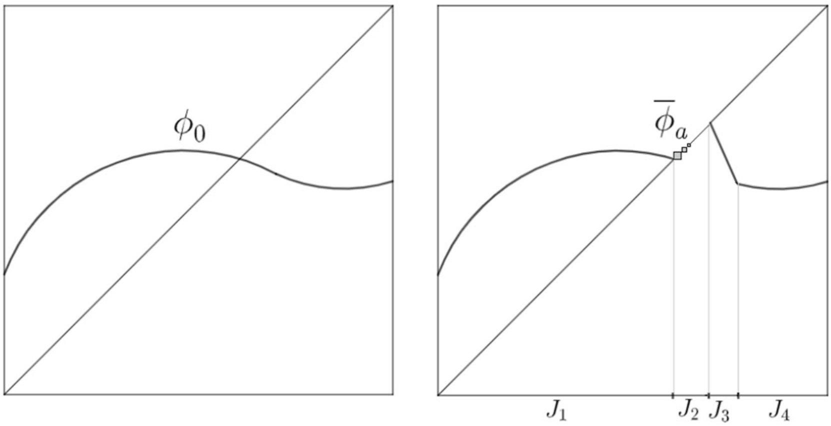


FIGURE 2. Graphs of ϕ_0 and ψ_a. $\overline{\phi}_a = T_2^{-1}\phi_a T_2$

in $C^0([0,1])$. Therefore, Corollary 3.4 can be applied for any map in $\mathcal{A}$, which is a dense subset of $C^0([0,1])$.

For $a, b \in (0,1]$, let $\phi_a, \phi_b \in C^0([0,1])$ be such that

$$\mathrm{mdim}_{\mathrm{M}}([0,1], d, \phi_a) = a \quad \text{and} \quad \mathrm{mdim}_{\mathrm{M}}([0,1], d, \phi_b) = b$$

(see Example 3.1). It follows from Theorem 3.7, item v, that

$$\mathrm{mdim}_{\mathrm{M}}([0,1]\times[0,1], d\times d, \phi_a\times\phi_b) = \mathrm{mdim}_{\mathrm{M}}([0,1], d, \phi_a) + \mathrm{mdim}_{\mathrm{M}}([0,1], d, \phi_b) = a+b.$$

Hence, we have:

Lemma 4.3. *Fix $n \in \mathbb{N}$. For any $a \in [0,n]$, there exists $\phi_a \in C^0([0,1]^n)$ such that*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, d^n, \phi_a) = \underline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, d^n, \phi_a) = a.$$

Furthermore, given that d^n (see (3.4)) and $\|\cdot\|$, where $\|(x_1,\ldots,x_n)\| = \sqrt{x_1^2+\cdots+x_n^2}$ for any $(x_1,\ldots,x_n) \in \mathbb{R}^n$, are uniformly equivalent, we have for any $a \in [0,n]$, there exists $\phi_a \in C^0(X^n)$ such that

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, \|\cdot\|, \phi_a) = \underline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, \|\cdot\|, \phi_a) = a.$$

Remark 4.4. Fix $r_1, r_2, \ldots, r_n \in (0,\infty)$ and $s_1, s_2, \ldots, s_n \in \mathbb{N}$ and for $i = 1, 2, \ldots, n$ take n maps $\phi_{s_i,r_i} \in C^0([0,1])$ defined as in Example 3.1. Thus for each $i = 1, 2, \ldots, n$ we have

$$\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi_{s_i,r_i}) = \frac{s_i}{r_i+s_i}.$$

From Theorem 3.7, item v, we have

$$\mathrm{mdim}_{\mathrm{M}}([0,1]^n, d^n, \phi_{s_1,r_1}\times\phi_{s_2,r_2}\times\cdots\times\phi_{s_n,r_n}) = \sum_{i=1}^{n} \frac{s_i}{r_i+s_i}.$$

Furthermore, it follows from 3.1 and Theorem 3.7 that for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathrm{mdim}_{\mathrm{M}}([0,1]^n, d^n, (\phi_{s_1,r_1} \times \cdots \times \phi_{s_n,r_n})^k) &= \mathrm{mdim}_{\mathrm{M}}([0,1]^n, d^n, (\phi_{s_1,r_1})^k \times \cdots \times (\phi_{s_n,r_n})^k) \\ &= \mathrm{mdim}_{\mathrm{M}}([0,1]^n, d^n, \phi_{ks_1,r_1} \times \cdots \times \phi_{ks_n,r_n}) \\ &= \sum_{i=1}^{n} \frac{ks_i}{r_i + ks_i}.\end{aligned}$$

Throughout this section, we will fix a compact riemannian manifold N with riemannian metric d and $\dim(N) = n \geq 1$. The proof of the following theorem consists in perturbing a map on small neighborhoods (on which we will work using coordinate charts) of the orbit of a periodic point, that is, on finitely many neighborhoods. Since the metric mean dimension depends on the metric, we must be careful to choose the charts that will be used to make the perturbations. For this reason we will take the charts given by the exponential map, which provides us the required properties. Indeed, for each $p \in N$, consider the *exponential map*

$$\exp_p : B_{\delta'}(0_p) \subseteq T_pN \to B_{\delta'}(p) \subseteq N,$$

where 0_p is the origin in the tangent space T_pN, δ' is the *injectivity radius* of N and $B_\epsilon(x)$ denote the open ball of radius $\epsilon > 0$ with center x. We will take $\delta_N = \frac{\delta'}{2}$. The exponential map has the following properties (see [6], Chapter III):

- Since N is compact, δ' does not depends on p.
- $\exp_p(0_p) = p$ and $\exp_p[B_{\delta_N}(0_p)] = B_{\delta_N}(p)$;
- $\exp_p : B_{\delta_N}(0_p) \to B_{\delta_N}(p)$ is a diffeomorphism;
- If $v \in B_{\delta_N}(0_p)$, taking $q = \exp_p(v)$ we have $d(p, q) = \|v\|$.
- The derivative of $\exp_p$ at the origin is the identity map:

$$D(\exp_p)(0) = \mathrm{id} : T_pN \to T_pN.$$

Since $\exp_p : B_{\delta_N}(0_p) \to B_{\delta_N}(p)$ is a diffeomorphism and $D(\exp_p)(0) = \mathrm{id} : T_pN \to T_pN$, we have $\exp_p : B_{\delta_N}(0_p) \to B_{\delta_N}(p)$ is a bi-Lipschitz map with Lipschitz constant close to 1. Therefore, we can assume that if $v_1, v_2 \in B_{\delta_N}(0_p)$, taking $q_1 = \exp_p(v_1)$ and $q_2 = \exp_p(v_2)$, we have $d(q_1, q_2) = \|v_1 - v_2\|$. Furthermore, we will identify $B_{\delta_N}(0_p) \subset T_pN$ with $B_{\delta_N}(0) = \{x \in \mathbb{R}^n : \|x\| < \delta_N\} \subseteq \mathbb{R}^n$.

Theorem 4.5. *For any $a \in [0, n]$, the set*

$$C_a(N) = \{\phi \in C^0(N) : \overline{\mathrm{mdim}}_{\mathrm{M}}(N, d, \phi) = \underline{\mathrm{mdim}}_{\mathrm{M}}(N, d, \phi) = a\}$$

is dense in $C^0(N)$.

Proof. Let $P^r(N)$ be the set consisting of C^r-differentiable maps on N with a periodic point. This set is C^0-dense in $C^0(N)$ (see [1,12]). Hence, in order to prove the theorem it is sufficient to show if $\phi_0 \in P^r(N)$, then for any $\varepsilon > 0$ there exists $\varphi_a \in C_a(N)$, with $d(\phi_0, \varphi_a) < \varepsilon$. Fix $\phi_0 \in P_r(N)$ and take $\varepsilon \in (0, \delta_N)$. Suppose that p is a periodic point of ϕ_0 with period k. We can

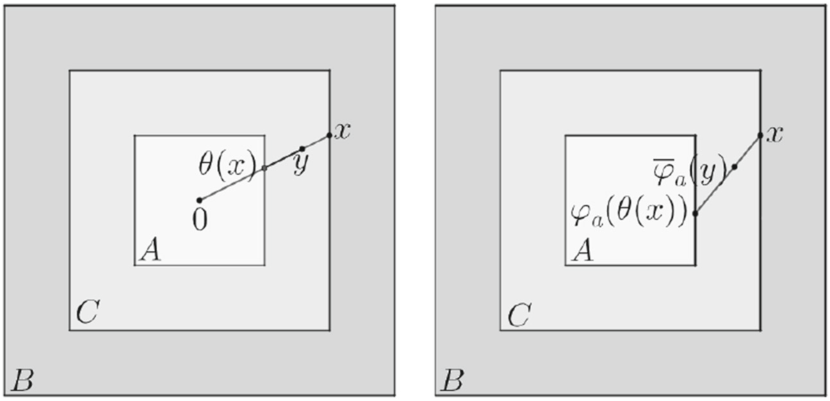


FIGURE 3. Extension of $\overline{\varphi}_a$

suppose that $B_\varepsilon(\phi_0^i(p)) \cap B_\varepsilon(\phi_0^j(p)) = \emptyset$, for $i, j = 1, \ldots, k$ with $i \neq j$. Take $\lambda \in (0, \varepsilon/4)$ such that $\phi_0(B_{4\lambda}(\phi_0^i(p))) \subseteq B_{\varepsilon/4}(\phi_0^{i+1}(p))$ for $i = 0, \ldots, k-1$. Take $\phi_a \in C^0([-\frac{\lambda}{3}, \frac{\lambda}{3}]^n)$ obtained by a cartesian product of maps given in Example 3.1, with $[-\frac{\lambda}{3}, \frac{\lambda}{3}]$ instead of $[0, 1]$, such that $\mathrm{mdim}_{\mathrm{M}}([-\frac{\lambda}{3}, \frac{\lambda}{3}]^n, \|\cdot\|, \phi_a) = a$ (see Lemma 4.3). Set

$$A = \left[-\frac{\lambda}{3}, \frac{\lambda}{3}\right]^n, \qquad B = [-\lambda, \lambda]^n \setminus (-2\lambda/3, 2\lambda/3)^n, \qquad C = [-\lambda, \lambda]^n \setminus (A \cup B).$$

Take the map $\varphi_a : A \cup B \to A \cup B$ defined by

$$\varphi_a(x) = \begin{cases} \phi_a(x), & \text{if } x \in A \\ x, & \text{if } x \in B. \end{cases}$$

Note that

$$\varphi_a(\partial A) = \partial A \quad \text{and} \quad \varphi_a\left(\partial\left([-2\lambda/3, 2\lambda/3]^n\right)\right) = \partial\left([-2\lambda/3, 2\lambda/3]^n\right).$$

Furthermore, if $(x_1, \ldots, x_n) \in \partial A$, then $x_i \in \{-\lambda/3, \lambda/3\}$ for some i and we have

$$\varphi_a(x_1, \ldots, x_i, \ldots, x_n) = (z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n), \tag{4.1}$$

for some $z_j \in \left[-\frac{\lambda}{3}, \frac{\lambda}{3}\right]$, for $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. Hence $(x_1, \ldots, x_i, \ldots, x_n)$ and $\varphi_a(x_1, \ldots, x_i, \ldots, x_n)$ belong to the same face of ∂A. Considering this fact, we extend φ_a to a continuous map $\bar{\varphi}_a : [-\lambda, \lambda]^n \to [-\lambda, \lambda]^n$.

For any $x \in \partial([-2\lambda/3, 2\lambda/3]^n)$, take the line segment passing through x and $0 \in \mathbb{R}^n$. This line passes through a unique point $\theta(x) \in \partial A$. Any $y \in C$ can be written as $y = t\theta(x) + (1-t)x$, for some $t \in [0, 1]$, where $x \in \partial([-2\lambda/3, 2\lambda/3]^n)$ (see Fig. 3, noting that from (4.1) we have $\theta(x)$ and $\varphi_a(\theta(x))$ belong to the same face of ∂A). We define

$$\bar{\varphi}_a(y) = \bar{\varphi}_a(t\theta(x) + (1-t)x) = t\varphi_a(\theta(x)) + (1-t)x, \qquad \text{for } y \in C.$$

We have $\bar{\varphi}_a : [-\lambda, \lambda]^n \to [-\lambda, \lambda]^n$ is a continuous map: For $y, z \in C$, there exist $x_y, x_z \in \partial([-2\lambda/3, 2\lambda/3]^n)$ and $t, s \in [0, 1]$ such that

$$y = t\theta(x_y) + (1-t)x_y \quad \text{and} \quad z = s\theta(x_z) + (1-s)x_z.$$

If y and z are close, then x_y, x_z are close and therefore $\theta(x_y)$ and $\theta(x_z)$ are close (note that $\theta : \partial([-2\lambda/3, 2\lambda/3]^n) \to \partial A$ is a continuous map), which implies that t and s are close. Given that φ_a is a continuous map, we have $\varphi_a(\theta(x_y))$ and $\varphi_a(\theta(x_z))$ and therefore $\bar{\varphi}_a(y)$ and $\bar{\varphi}_a(z)$ are close. Next, if $y \in \partial A$, then $t = 1$ and thus $y = \theta(x)$. Therefore

$$\bar{\varphi}_a(y) = \bar{\varphi}_a(\theta(x)) = \varphi_a(\theta(x)).$$

If $y \in \partial([-2\lambda/3, 2\lambda/3]^n)$, then $t = 0$. Thus $y = x$ and therefore

$$\bar{\varphi}_a(y) = \bar{\varphi}_a(x) = x = \varphi_a(x).$$

From 4.1 we have if $t \in [0, 1]$ then $t\theta(x) + (1-t)x$ and $\varphi_a(t\theta(x) + (1-t)x)$ belong to C. Given that $\bar{\varphi}_a(\partial C) = \partial C$ ($\bar{\varphi}_a$ is the identity on ∂B and it is equal to φ_a on ∂A, which is surjective), we have by the continuity of $\bar{\varphi}_a$ that $\bar{\varphi}_a(C) = C$. Therefore, $\text{mdim}_{\text{M}}(C, \|\cdot\|, \bar{\varphi}_a) \leq a$. Hence,

$$\begin{aligned} \text{mdim}_{\text{M}}([-\sigma, \sigma]^n, \|\cdot\|, \bar{\varphi}_a) &= \max\{\text{mdim}_{\text{M}}(A, \|\cdot\|, \bar{\varphi}_a), \text{mdim}_{\text{M}}(B \cup C, \|\cdot\|, \bar{\varphi}_a)\} \\ &= \text{mdim}_{\text{M}}(A, \|\cdot\|, \bar{\varphi}_a) = \text{mdim}_{\text{M}}(A, \|\cdot\|, \phi_a) = a. \end{aligned}$$

Consider

$$\psi_a(q) = \begin{cases} \exp_{\phi_0^{i+1}(p)} \circ \bar{\varphi}_a \circ \exp^{-1}_{\phi_0^i(p)}(q), & \text{if } q \in N_i = \exp_{\phi_0^i(p)}([-\lambda, \lambda]^n), \text{ for some } i \\ \phi_0(q), & \text{if } q \in B = N \setminus \left(\bigcup_{i=1,\ldots,k} \exp_{\phi_0^i(p)}((-2\lambda, 2\lambda)^n) \right). \end{cases}$$

Next, we extend ψ_a to a continuous map $\bar{\psi}_a : N \to N$. For any $u \in \partial([-2\lambda, 2\lambda]^n)$, take the line segment passing through u and $0 \in \mathbb{R}^n$. This line passes through a unique point $\beta(u) \in \partial[-\lambda, \lambda]^n$. Any $w \in C_i = \exp_{\phi_0^i(p)}[[-2\lambda, 2\lambda]^n \setminus [-\lambda, \lambda]^n]$ can be written as

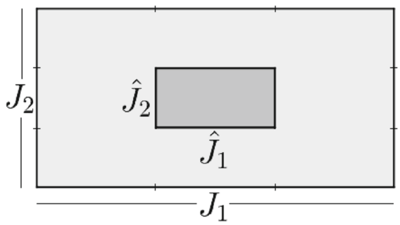
$$w = \exp_{\phi_0^i(p)}(t\beta(u) + (1-t)u),$$

for some $t \in [0, 1]$, where $u \in \partial([-2\lambda, 2\lambda]^n)$. For $w \in C_i$ we set

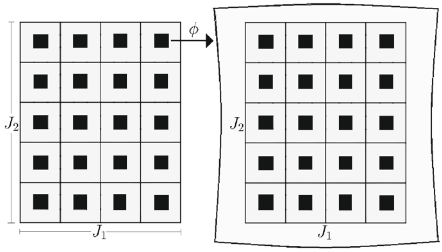
$$\begin{aligned} \bar{\psi}_a(w) &= \bar{\psi}_a(\exp_{\phi_0^i(p)}(t\beta(u) + (1-t)u)) \\ &= \exp_{\phi_0^{i+1}(p)}[t\bar{\varphi}_a(\beta(u)) + (1-t)\exp^{-1}_{\phi_0^{i+1}(p)}(\phi_0(\exp_{\phi_0^i(p)}(u)))] \\ &= \exp_{\phi_0^{i+1}(p)}[t\beta(u) + (1-t)\exp^{-1}_{\phi_0^{i+1}(p)}(\phi_0(\exp_{\phi_0^i(p)}(u)))]. \end{aligned}$$

We have $\bar{\psi}_a : N \to N$ is a continuous map (note that $\beta : \partial([-2\lambda, 2\lambda]^n) \to \partial[-\lambda, \lambda]^n$ is a continuous map). Furthermore, we have $\hat{d}(\phi_0, \bar{\psi}_a) < \varepsilon$. Note if $q \in N_i$, we have

$$\begin{aligned} (\overline{\psi}_a)^s(q) &= \exp_{\phi_0^{(i+s) \bmod k}(p)} \circ (\bar{\varphi}_a)^s \circ \exp^{-1}_{\phi_0^i(p)}(q) \quad \text{and} \\ (\overline{\psi}_a)^k(q) &= \exp_{\phi_0^i(p)} \circ (\bar{\varphi}_a)^k \circ \exp^{-1}_{\phi_0^i(p)}(q). \end{aligned}$$



(a) $J^2 = J_1 \times J_2$. $\hat{J}^2 = \hat{J}_1 \times \hat{J}_2$



(b) A strong $(2, \epsilon, 20)$-horseshoe

FIGURE 4. Strong horseshoe

Hence, $A \subseteq [-\lambda, \lambda]^n$ is an $(s, \overline{\varphi}_a, \epsilon)$-separated set if and only if $\exp_{\phi_0^i(p)}(A) \subseteq N$ is an $(s, \overline{\psi}_a, \epsilon)$-separated set for any $\epsilon > 0$. Therefore, setting $L = \bigcup_{i=1}^{k} N_i$, we have

$$\begin{aligned} \operatorname{sep}(s, \overline{\psi}_a|_L, \epsilon) &= k \operatorname{sep}(s, \overline{\varphi}_a, \epsilon) \quad \text{and thus} \\ \operatorname{mdim}_{\mathrm{M}}(L, d, \bar{\psi}_a|_L) &= \operatorname{mdim}_{\mathrm{M}}([-\lambda, \lambda]^n, \|\cdot\|, \bar{\varphi}_a). \end{aligned}$$

Set $K = \cup_{i=0}^{\infty} \bar{\psi}_a^{-i}(L)$ and $Z = K^c$. Note that $\Omega(\bar{\psi}_a|_K) \subseteq L$ and $\phi_0|_Z$ is a differentiable map. Hence $\operatorname{mdim}_{\mathrm{M}}(Z, d, \bar{\psi}_a|_Z) = 0$ and therefore

$$\begin{aligned} \operatorname{mdim}_{\mathrm{M}}(N, d, \bar{\psi}_a) &= \max\{\operatorname{mdim}_{\mathrm{M}}(K, d, \bar{\psi}_a|_K), \operatorname{mdim}_{\mathrm{M}}(Z, d, \bar{\psi}_a|_Z)\} \\ &= \max\{\operatorname{mdim}_{\mathrm{M}}(L, d, \bar{\psi}_a|_L), \operatorname{mdim}_{\mathrm{M}}(Z, d, \bar{\psi}_a|_Z)\} \\ &= \operatorname{mdim}_{\mathrm{M}}(L, d, \bar{\psi}_a|_L) = a, \end{aligned}$$

which proves the theorem. $\square$

In [4], Theorem A, the authors proved if $\dim(N) \geq 2$, then the set consisting of homeomorphisms with upper metric mean dimension equal to $n = \dim(N)$ is residual in $\operatorname{Hom}(N)$. Furthermore, they showed the set consisting of continuous maps on $[0, 1]$ with upper metric mean dimension equal to 1 is residual in $C^0([0, 1])$. Inspired by the proof of these facts, we will show the set consisting of continuous maps on N with upper metric mean dimension equal to n, which we will denote by $\overline{C}_n(N)$, is residual in $C^0(N)$.

A closed n-rectangular box is a product $J^n = J_1 \times \cdots \times J_n$ of closed subintervals J_i for any $i = 1, \ldots, n$. From now on, we denote by J^n a closed n-rectangular box and we set

$$|J^n| := \min_{i=1,\ldots,n} |J_i|, \quad \text{where} \quad J^n = J_1 \times \cdots \times J_n.$$

For any closed interval $J = [a, b]$, let $\hat{J} = [\frac{2a+b}{3}, \frac{a+2b}{3}]$, that is, the second third of J. For a closed n-rectangular box $J^n = J_1 \times \cdots \times J_n$, set $\hat{J}^n = \hat{J}_1 \times \cdots \times \hat{J}_n$ (see Fig. 4a).

For $\epsilon \in (0,1)$ and $k \in \mathbb{N}$, we say a closed n-rectangular box $J^n \subset U \subset \mathbb{R}^n$ is a *strong* (n,ϵ,k)*-horseshoe* of a continuous map $\phi : U \to \mathbb{R}^n$ if $|J^n| > \epsilon$ and J^n contains k closed n-rectangular boxes $J_1^n, \dots, J_k^n \subseteq J^n$, with $(J_s^n)^\circ \cap (J_r^n)^\circ = \emptyset$ for $s \neq r$, such that $|J_i^n| > \frac{|J^n|}{2\sqrt[n]{k}}$ and $J^n \subset (\phi(\hat{J}_i^n))^\circ$ for any $i = 1, \dots, k$. In Fig. 4b we present an example of a strong $(2,\epsilon,20)$-horseshoe.

We say $\phi \in C^0(N)$ has a strong (n,ϵ,k)-horseshoe J^n, where $J^n \subset \mathbb{R}^n$ is a closed n-rectangular box, if there exist s exponential charts $\exp_i : B(0,\delta_N) \to N$, for $i = 1, \dots, s$, such that:

- $\phi_i = \exp_{(i+1) \bmod s} \circ \phi \circ \exp_i^{-1} : B(0,\delta) \to B(0,\delta_N)$ is well defined for some $\delta \leq \delta_N$;
- $J^n \subset (\phi_i(J^n))^\circ$ for each $i = 1, \dots, s$;
- J^n is a strong (n,ϵ,k)-horseshoe for ϕ_i for each $i = 1, \dots, s$.

To simplify the notation, we will set $\phi_i = \phi$ for each $i = 1, \dots, s$.

For $\epsilon > 0$ and $k \in \mathbb{N}$, set

$$H(n,\epsilon,k) = \{\phi \in C^0(N) : \phi \text{ has a strong } (n,\epsilon,k)\text{-horseshoe}\}$$
$$H(n,k) = \bigcup_{i \in \mathbb{N}} H\left(n, \frac{1}{i^2}, 3^{n\,k\,i}\right)$$
$$\mathcal{H}^n = \bigcap_{k=1}^{\infty} H(n,k).$$

Theorem 4.6. *$\mathcal{H}^n$ is residual and if $\phi \in \mathcal{H}^n$, then $\overline{\mathrm{mdim}}_{\mathrm{M}}(N,d,\phi) = n$. Therefore, for any $n \geq 1$, if N is a n-dimensional compact riemannian manifold with riemannian metric d, the set $\overline{C}_n(N) = \{\phi \in C^0(N) : \overline{\mathrm{mdim}}_{\mathrm{M}}(N,d,\phi) = n\}$ is residual in $C^0(N)$.*

Proof. We prove for any $\epsilon \in (0,\delta_N)$ and $k \in \mathbb{N}$, we have $H(n,\epsilon,k)$ is nonempty. In fact, consider the map $g : [0,1] \to [0,1]$ defined in Example 3.1. For any $s \geq 2$, g^s has a strong $\left(1, 1 - \frac{4}{3^s}, \frac{3^s-3}{3}\right)$-horseshoe (see Fig. 5):

$$J = \left[\frac{1}{3^s}, \frac{4}{3^s}\right] \cup \left[\frac{4}{3^s}, \frac{7}{3^s}\right] \cup \dots \cup \left[\frac{3^s-5}{3^s}, \frac{3^s-2}{3^s}\right], \quad |J| = 3^s - 3 > 1 - \frac{4}{3^s},$$
$$|J_r| := \left|\left[\frac{3(r-1)+1}{3^s}, \frac{3(r-1)+4}{3^s}\right]\right| = \frac{3}{3^s} > \frac{1}{2|J|} = \frac{1}{2(3^s-3)},$$
$$J \subset (0,1) = (g^s(\hat{J}_r))^\circ = \left(g^s\left(\left[\frac{3(r-1)+2}{3^s}, \frac{3(r-1)+3}{3^s}\right]\right)\right)^\circ,$$

for any$r = 1, \dots, \frac{3^s-3}{3}$.

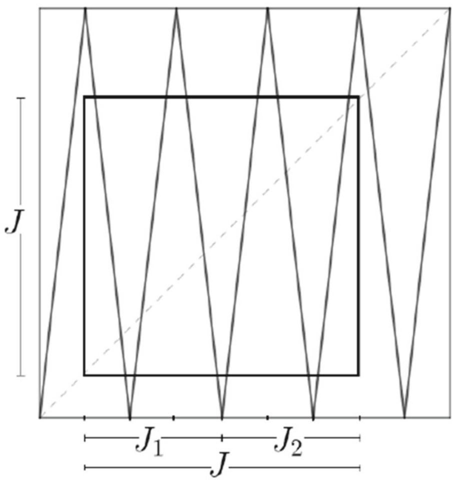


Figure 5. J is a strong (1,1-4/9,2)-horseshoe for g^2

Take an s large enough such that $1-\frac{4}{3^s} > \epsilon$ and $3^s-3 \geq k$. We have J^n is a strong (n,ϵ,k)-horseshoe of $\tilde{g} := g\times\cdots\times g \in C^0([0,1]^n)$. We can make a affine change of variable and we can assume that $g : [-\delta_N, \delta_N] \to [-\delta_N, \delta_N]$. Let $\psi : B(0,\delta_N) \to N$ be an exponential chart. The map $\psi\circ\tilde{g}\circ\psi^{-1} : \psi([-\delta_N,\delta_N]^n) \to \psi([-\delta_N,\delta_N]^n)$ can be extended to a continuous map $\hat{g}$ on N as we made in Theorem 4.5 (note $\tilde{g}$ has the properties needed in order to do this extension). We have $\hat{g} \in H(n,\epsilon,k)$.

$H(n,\epsilon,k)$ is open in $C^0(N)$: if $\phi \in H(n,\epsilon,k)$ and J^n is a strong (n,ϵ,k)-horseshoe of ϕ we can take a small enough open neighborhood U of ϕ such that for any $\psi \in U$ we have J^n is a strong (n,ϵ,k)-horseshoe of ψ.

$H(n,k)$ is dense in $C^0(N)$: fix $\psi \in C^0(N)$ with a s-periodic point. Every small neighborhood of the orbit of this point can be perturbed in order to obtain a strong $\left(n,\frac{1}{i^2},3^{n\,k\,i}\right)$ horseshoe for a ϕ close to ψ for a large enough i (see the proof of Theorem 4.5).

The above facts prove that $\mathcal{H}^n = \bigcap_{k=1}^{\infty} H(n,k)$ is residual in $C^0(N)$.

Finally, we prove $\overline{\text{mdim}}_{\text{M}}(N,d,\phi) = n$ for any $\phi \in \mathcal{H}^n$. Take $\phi \in \mathcal{H}^n$. We have $\phi \in H(n,k)$ for any $k \geq 1$. Therefore, for any $k \in \mathbb{N}$, there exists i_k, with $i_k < i_{k+1}$, such that ϕ has a strong $\left(n,\frac{1}{i_k^2},3^{n\,k\,i_k}\right)$-horseshoe $J^n_{i_k}$, consisting of $3^{n\,k\,i_k}$ rectangular boxes $J^n(i_k,1),\ldots,J^n(i_k,3^{n\,k\,i_k})$, such that $J^n \subset (\phi_i(\hat{J}^n(i_k,t)))^\circ$ for each $t=1,\ldots,3^{n\,k\,i_k}$, where $\phi_i = \exp_{(i+1)\bmod s}\circ\phi\circ\exp_i^{-1}$. For each $k\in\mathbb{N}$, set $\varepsilon_k = \frac{1}{4i_k^2 3^{k\,i_k}}$. For any $m \in \mathbb{N}$, set

$$C_{n,k}(t_0,t_1,\ldots,t_{m-1}) = \{x \in J^n_{i_k} : \phi^l(x) \in \hat{J}^n_{i_k,t_l} \text{ for all } l \in \{0,\ldots,m-1\}\}.$$

From the definition, we have $|J^n_{i_k,t}| > \varepsilon_k$, for each $t = 1, \ldots, 3^{n\,k\,i_k}$. Thus, the set consisting of any point on each $\tilde{C}_{n,k} = \exp[C_{n,k}(t_0, t_1, \ldots, t_{m-1})]$ is an (m, ϕ, ε_k) separated set. Therefore, for each $m \in \mathbb{N}$ we have

$$\operatorname{sep}(m, \phi, \varepsilon_k) \geq \left(3^{n\,k\,i_k}\right)^m \quad \text{and hence} \quad \frac{\operatorname{sep}(\phi, \varepsilon_k)}{|\log \varepsilon_k|} \geq n \frac{\log 3^{k\,i_k}}{\log 3^{k\,i_k} + \log 4 i_k^2}.$$

Note $\frac{\log 3^{k\,i_k}}{\log 3^{k\,i_k} + \log 4 i_k^2} \to 1$ as $k \to \infty$. This fact implies that

$$\overline{\operatorname{mdim}}_{\mathrm{M}}(N, d, \phi) \geq n,$$

which proves the theorem, since for any $\psi \in C^0(N)$, the inequality $\overline{\operatorname{mdim}}_{\mathrm{M}}(N, d, \psi) \leq n$ always hold. □

The continuity of the topological entropy is one of the most studied problem in dynamical systems (see [2,19,24]). If X is the interval or the circle, Block, in [2], proved the topological entropy map is not continuous on continuous maps on X with finite topological entropy. Now, Yano in [24] proved the topological entropy map is continuous on any continuous map $\phi \in C^0(N)$ with infinite topological entropy. For the metric mean dimension, it follows from Theorem 4.5 that:

Corollary 4.7. *If N is any compact riemannian manifold with riemannian metric d, then* $\operatorname{mdim}_{\mathrm{M}} : \mathrm{C}^0(\mathrm{N}) \to \mathbb{R}$ *is not continuous anywhere.*

A real valued function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is called *lower* (respectively *upper*) *semi-continuous on a point* $x \in X$ if

$$\liminf_{y \to x} \varphi(y) \geq \varphi(x) \quad (\text{respectively } \limsup_{y \to x} \varphi(y) \leq \varphi(x)).$$

φ is called *lower* (respectively *upper*) *semi-continuous* if is lower (respectively upper) semi-continuous on any point of X.

The map $h_{\mathrm{top}} : C^0([0,1]) \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous (see [18], Corollary 1). However, for metric mean dimension we have if $X = [0,1]$ or $\mathbb{S}^1$, then $\operatorname{mdim}_{\mathrm{M}} : C^0(X) \to \mathbb{R}$ is nor lower neither upper semi-continuous (see [20], Proposition 7.6). Furthermore, from Theorem 4.5 we have:

Corollary 4.8. *Let N be any compact riemannian manifold with riemannian metric d. We have* $\operatorname{mdim}_{\mathrm{M}} : C^0(N) \to \mathbb{R}$ *is nor lower neither upper semi-continuous on maps with metric mean dimension in* $(0, \dim(N))$*. Furthermore,* $\operatorname{mdim}_{\mathrm{M}} : C^0(N) \to \mathbb{R}$ *is not lower semi-continuous on maps with metric mean dimension in* $(0, \dim(N)]$ *and is not upper semi-continuous on maps with metric mean dimension in* $[0, \dim(N))$.

5. Density of Continuous Maps on Cantor Sets with Positive Metric Mean Dimension

Bobok and Zindulka in [3] shown that if X is an uncountable compact metrizable space of topological dimension zero, then given any $a \in [0, \infty]$ there exists a homeomorphism on X whose topological entropy is a. In particular, there exist homeomorphisms on the Cantor set with infinite topological entropy. We will use the techniques presented by Bobok and Zindulka in order to prove there exist infinitely many continuous maps on the Cantor set with positive metric mean dimension. In fact, any $x \in [0, 1]$ is written in base 3 as

$$x = \sum_{n=1}^{\infty} x_n 3^{-n} \quad \text{where } x_n \in \{0, 1, 2\}.$$

A number x belongs to the ternary Cantor set if and only if it has a ternary representation where the digit one does not appear. Therefore, we can consider

$$\boldsymbol{C} = \{(x_1, x_2, \dots) : x_n = 0, 2 \text{ for } n \in \mathbb{N}\} = \{0, 2\}^{\mathbb{N}} \tag{5.1}$$

as being the Cantor set endowed with the metric

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{n=1}^{\infty} 3^{-n} |x_n - y_n| = \left| \sum_{n=1}^{\infty} x_n 3^{-n} - \sum_{n=1}^{\infty} y_n 3^{-n} \right|. \tag{5.2}$$

Proposition 5.1. *For each $j \in \mathbb{N}$, there exists $\psi_j \in C^0(\boldsymbol{C})$ with*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j) = \overline{\mathrm{mdim}}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j) = \frac{j \log 2}{(j+1) \log 3}.$$

Proof. For any $k \geq 1$, set

$$\boldsymbol{C}_k = \{(x_i)_{i=1}^{\infty} : x_i = 0 \text{ for } i \leq k-1, x_k = 2 \text{ and } x_i \in \{0, 2\} \text{ for } i \geq k+1\}.$$

Note that if $k \neq s$, then $\boldsymbol{C}_k \cap \boldsymbol{C}_s = \emptyset$ and $\boldsymbol{C} \setminus \cup_{k=1}^{\infty} \boldsymbol{C}_k = \{(0, 0, \dots)\}$. Furthermore, each $\boldsymbol{C}_k$ is a clopen subset homeomorphic to $\boldsymbol{C}$ via the homeomorphism

$$T_k : \boldsymbol{C}_k \to \boldsymbol{C}, \quad (\underbrace{0, \dots, 0}_{(k-1)\text{-times}}, 2, x_1, x_2, \dots) \mapsto (x_1, x_2, \dots),$$

which is Lipschitz. For $j \in \mathbb{N}$, take $\psi_j \in C^0(\boldsymbol{C})$ the map defined as $\psi_j(0, 0, \dots) = (0, 0, \dots)$ and $\psi_j|_{\boldsymbol{C}_k} = T_k^{-1} \sigma^{jk} T_k$ for $k \geq 1$. It is not difficult to prove that ψ_j is a continuous map. Take $\varepsilon > 0$. For any $k \geq 1$, set $\varepsilon_k = 3^{-k(j+1)}$. There exists $k \geq 1$ such that $\varepsilon \in [\varepsilon_{k+1}, \varepsilon_k]$. For $n \geq 1$ and $k \geq 1$, take $\bar{z}_1 = (z_1^1, \dots, z_{jk}^1), \dots, \bar{z}_n = (z_1^n, \dots, z_{jk}^n)$, with $z_i^s \in \{0, 2\}$, and set

$$A^k_{\bar{z}_1, \dots, \bar{z}_n} = \{(\underbrace{0, \dots, 0}_{(k-1)\text{-times}}, 2, z_1^1, \dots, z_{jk}^1, \dots, z_1^n, \dots, z_{jk}^n, x_1, \dots, x_s, \dots) : x_i \in \{0, 2\}\} \subseteq \boldsymbol{C}_k.$$

Note that if $A^k_{\bar{z}_1,\dots,\bar{z}_n} \neq A^k_{\bar{w}_1,\dots,\bar{w}_n}$ and $\bar{x} \in A^k_{\bar{z}_1,\dots,\bar{z}_n}$, $\bar{y} \in A^k_{\bar{w}_1,\dots,\bar{w}_n}$, then $d_{n+1}(\bar{x},\bar{y}) > \frac{1}{3^{k(j+1)}}$, where d_{n+1} is considered with respect to ψ_j. Therefore $\operatorname{sep}(n+1,\psi_j,\varepsilon_k) \geq 2^{jnk}$ and hence

$$\limsup_{n\to\infty} \frac{\log \operatorname{sep}(n+1,\psi_j,\varepsilon)}{n+1} \geq \limsup_{n\to\infty} \frac{\log \operatorname{sep}(n+1,\psi_j,\varepsilon_k)}{n+1}$$
$$\geq \lim_{n\to\infty} \frac{n\log(2^{jk})}{n+1} = \log 2^{jk},$$

thus

$$\underline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) \geq \lim_{k\to\infty} \frac{\log \operatorname{sep}(\psi_j,\varepsilon_k)}{-\log \varepsilon_{k+1}} \geq \lim_{k\to\infty} \frac{\log(2^{jk})}{\log(3^{(k+1)(j+1)})}$$
$$= \lim_{k\to\infty} \frac{kj\log 2}{(k+1)(j+1)\log 3}$$
$$= \frac{j\log 2}{(j+1)\log 3}.$$

Therefore

$$\overline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) \geq \underline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) \geq \frac{j\log 2}{(j+1)\log 3}. \tag{5.3}$$

On the other hand, note that for each $l \in \{1,\dots,k\}$, the sets $A^l_{\bar{z}_1,\dots,\bar{z}_n}$ have d_n-diameter less than ε_k. Furthermore, the sets $\{(0,0,\dots)\}$ and $\bigcup_{s=k+1}^{\infty} \boldsymbol{C}_s$ has d_n-diameter less than ε_k. Hence

$$\operatorname{cov}(n,\psi_j,\varepsilon_k) \leq k2^{njk} + 2 \leq 2k2^{njk}$$

and therefore

$$\operatorname{cov}(\psi_j,\varepsilon_k) \leq \lim_{n\to\infty} \frac{\log(2k2^{njk})}{n} = \log 2^{jk}.$$

Hence

$$\overline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) = \limsup_{\varepsilon\to 0} \frac{\operatorname{cov}(\psi_j,\varepsilon)}{-\log\varepsilon} \leq \limsup_{k\to\infty} \frac{\operatorname{cov}(\psi_j,\varepsilon_{k+1})}{-\log\varepsilon_k} \leq \frac{j\log 2}{(j+1)\log 3}. \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$\underline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) = \overline{\operatorname{mdim}}_{\mathrm{M}}(\boldsymbol{C},d,\psi_j) = \frac{j\log 2}{(j+1)\log 3},$$

which proves the proposition. □

For any continuous map $\phi : X \to X$ we always have

$$\overline{\operatorname{mdim}}_{\mathrm{M}}(X,d,\phi) \leq \overline{\dim}_{\mathrm{B}}(X,d) \quad \text{and} \quad \underline{\operatorname{mdim}}_{\mathrm{M}}(X,d,\phi) \leq \underline{\dim}_{\mathrm{B}}(X,d).$$

Therefore $\operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C},d,\phi) \leq \dim_{\mathrm{B}}(\boldsymbol{C}) = \frac{\log 2}{\log 3}$ for any continuous map $\phi : \boldsymbol{C} \to \boldsymbol{C}$. A question that arises from the above proposition is: is there any $\phi \in C^0(\boldsymbol{C})$ with $\operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C},d,\phi) = \frac{\log 2}{\log 3}$?

Remark 5.2. Consider ψ_j as in Proposition 5.1. Note that $\psi_{sj} = \psi_j^s$ for any $s \in \mathbb{N}$. It follows from the proposition that

$$\operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j^s) = \operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_{sj}) = \frac{sj \log 2}{(sj+1)\log 3} = \frac{s \log 2}{(s+\frac{1}{j})\log 3}.$$

Therefore, for any $s \in \mathbb{N}$, we have

$$\operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j) < \operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j^s) < s \operatorname{mdim}_{\mathrm{M}}(\boldsymbol{C}, d, \psi_j).$$

For any $m \geq 2$, take $X_m = \{1, 2, \ldots, m\}$. We endow $X_m^{\mathbb{K}}$ with the metric d given in 5.2. It follows from Proposition 5.1 there exist continuous maps on $X_m^{\mathbb{K}}$ with positive metric mean dimension.

Theorem 5.3. *Take $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. If $C_a = \{\phi \in C^0(X_m^{\mathbb{K}}) : \operatorname{mdim}_{\mathrm{M}}(X_m^{\mathbb{K}}, d, \phi) = a\} \neq \emptyset$, then C_a is dense in $C^0(X_m^{\mathbb{K}})$.*

Proof. We will prove the case $\mathbb{K} = \mathbb{N}$ (the case $\mathbb{K} = \mathbb{Z}$ is analogous). We will fix a continuous map $\phi : X_m^{\mathbb{N}} \to X_m^{\mathbb{N}}$, given by $\phi(x_1, x_2, \ldots) = (y_1(\bar{x}), y_2(\bar{x}), \ldots)$, for any $\bar{x} = (x_1, x_2, \ldots) \in X_m^{\mathbb{N}}$. We will approximate ϕ by a sequence of continuous maps in C_a.

Firstly, we prove that C_0 is dense in $C^0(X_m^{\mathbb{N}})$. Consider the sequence of continuous maps on $X_m^{\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$, defined by

$$\phi_n(\bar{x}) = (y_1(\bar{x}), y_2(\bar{x}), \ldots, y_n(\bar{x}), x_0, x_0, \ldots) \quad \text{for any } n \in \mathbb{N} \text{ and some } x_0 \in X_m. \tag{5.5}$$

Since the image of ϕ_n is a finite set, then we have $\operatorname{mdim}_{\mathrm{M}}(X_m^{\mathbb{N}}, d, \phi_n) = 0$ for any $n \in \mathbb{N}$. Note that ϕ_n converges uniformly to ϕ as $n \to \infty$. This fact proves the set C_0 is dense in $C^0(X_m^{\mathbb{N}})$.

Next, fix $a > 0$ and suppose that $C_a \neq \emptyset$. Since C_0 is dense in $C^0(X_m^{\mathbb{N}})$, in order to prove that C_a is dense in $C^0(X_m^{\mathbb{N}})$ we can prove that any map in C_0 can be approximate by a sequence of maps in C_a. Therefore, we can suppose that $\phi \in C_0$ is a map as the given in (5.5), that is, for any $\bar{x} = (x_1, x_2, \ldots) \in X_m^{\mathbb{N}}$,

$$\phi(\bar{x}) = (y_1(\bar{x}), y_2(\bar{x}), \ldots, y_K(\bar{x}), z_0, z_0, \ldots) \quad \text{for some } K \in \mathbb{N} \text{ and some } z_0 \in X_m.$$

Suppose that $\psi_a \in C_a$ is given by

$$\psi_a(\bar{x}) = (z_1(\bar{x}), z_2(\bar{x}), \ldots) \qquad \text{for any } \bar{x} = (x_1, x_2, \ldots) \in X_m^{\mathbb{N}}.$$

For each $n \geq K+1$, set $\bar{x}^n = (x_{n+1}, x_{n+2}, \ldots)$. Consider the sequence of continuous maps on $X_m^{\mathbb{N}}$, $(\phi_n)_{n\geq K+1}$, where

$$\phi_n(\bar{x}) = (y_1(\bar{x}), y_2(\bar{x}), \ldots, y_K(\bar{x}), \underbrace{z_0, \ldots, z_0}_{(n-K)\text{-times}}, z_1(\bar{x}^n), z_2(\bar{x}^n), \ldots)$$
$$\text{for any } n \geq K+1 \text{ and } \bar{x} \in X_m^{\mathbb{N}}.$$

We have ϕ_n converges uniformly to ϕ as $n \to \infty$. Note that

$$\sum_{j=n}^{\infty} 3^{-j}|x_{j-n+1} - y_{j-n+1}| = 3^{1-n}\sum_{j=n}^{\infty} 3^{-j+n-1}|x_{j-n+1} - y_{j-n+1}|$$
$$= 3^{1-n}\sum_{j=1}^{\infty} 3^{-j}|x_j - y_j|. \tag{5.6}$$

Next, fix $n \in \mathbb{N}$ and take $\varepsilon > 0$. For any $p \geq n \in \mathbb{N}$, let A be a (p, ψ_a, ε)-separated set. Take

$$\tilde{A} = \{(x_i)_{i\in\mathbb{N}} : x_j = z_0 \text{ for } 1 \leq j \leq n, (x_{n+i})_{i\in\mathbb{N}} \in A\} = \underbrace{\{z_0\} \times \cdots \times \{z_0\}}_{n\text{-times}} \times A.$$

Note that if $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are two different sequences in A, from (5.6) we have

$$d_p^{\phi_n}((z_0, \ldots, z_0, x_1, x_2, \ldots), (z_0, \ldots, z_0, y_1, y_2, \ldots))$$
$$\geq 3^{1-n} d_p^{\psi_a}((x_1, \ldots), (y_1, \ldots)) \geq 3^{1-n}\varepsilon.$$

Hence $\tilde{A}$ is a $(p, \phi_n, 3^{1-n}\varepsilon)$-separated set. Therefore $\mathrm{sep}(\psi_a, \varepsilon) \leq \mathrm{sep}(\phi_n, 3^{1-n}\varepsilon)$ and thus

$$\limsup_{\varepsilon\to 0} \frac{\mathrm{sep}(\psi_a, \varepsilon)}{|\log \varepsilon|} \leq \limsup_{\varepsilon\to 0} \frac{\mathrm{sep}(\phi_n, 3^{1-n}\varepsilon)}{|\log 3^{1-n} + \log \varepsilon|} = \limsup_{\varepsilon\to 0} \frac{\mathrm{sep}(\phi_n, 3^{1-n}\varepsilon)}{|\log 3^{1-n}\varepsilon|},$$

which proves that $\overline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}}, d, \psi_a) \leq \overline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}}, d, \phi_n)$.

On the other hand, note that

$$\Omega(\phi_n) \subseteq \underbrace{X_m \times \cdots \times X_m}_{K\text{-times}} \times \underbrace{\{z_0\} \times \cdots \times \{z_0\}}_{(n-K)\text{-times}} \times X_m \times X_m \times \cdots := Z,$$

where $\Omega(\varphi)$ is the non-wandering set of a continuous map φ. Hence, we can consider the restriction $\phi_n|_Z : Z \to Z$ in order to find the metric mean dimension of ϕ_n. Take $\varepsilon < 3^{-n}$ small enough such that if $d(\bar{x}, \bar{y}) < \varepsilon$, then $d(\phi(\bar{x}), \phi(\bar{y})) < 3^{-K}$. Let B be a (p, ψ_a, ε)-spanning set and C a (p, ϕ, ε)-spanning set. Set

$$\tilde{C} = \{(x_1, \ldots, x_K) : (x_i)_{i\in\mathbb{N}} \in C\} \quad \text{and} \quad \tilde{B} = \tilde{C} \times \underbrace{\{z_0\} \times \cdots \times \{z_0\}}_{(n-K)\text{-times}} \times B.$$

Take any $\bar{y} = (y_1, y_2, \ldots, y_K, z_0, \ldots, z_0, y_{n+1}, y_{n+2}, \ldots) \in Z$. There exists

$$\bar{a} = (y_1, y_2, \ldots, y_K, z_0, \ldots, z_0, a_{n+1}, a_{n+2}, \ldots) \in C$$

with $d_p^{\phi}(\bar{y}, \bar{a}) < \varepsilon$ ($\bar{a}$ has this form because $\varepsilon < 3^{-n}$). Set

$$\tilde{x} = (y_1, y_2, \ldots, y_K, z_0, \ldots, z_0, x_{n+1}, x_{n+2}, \ldots),$$

for some $(x_{n+1}, x_{n+2}, \ldots) \in B$ with $d_p^{\psi_a}((y_{n+1}, y_{n+2}, \ldots), (x_{n+1}, x_{n+2}, \ldots)) < \varepsilon$. In particular $d((y_{n+1}, y_{n+2}, \ldots), (x_{n+1}, x_{n+2}, \ldots)) < \varepsilon$. Note that $\tilde{x} \in \tilde{B}$. Hence,

$$\begin{aligned} d(\bar{y},\tilde{x}) &= d((y_1,\ldots,y_K,z_0,\ldots,z_0,y_{n+1},y_{n+2},\ldots),\\ &\quad (y_1,\ldots,y_K,z_0,\ldots,z_0,x_{n+1},x_{n+2},\ldots))\\ &= \sum_{i=n+1}^{\infty} 3^{-i}|y_i - x_i| = 3^{-n} \sum_{i=n+1}^{\infty} 3^{-i+n}|y_i - x_i|\\ &= 3^{-n}\sum_{i=1}^{\infty} 3^{-i}|y_{n+i} - x_{n+i}|\\ &= 3^{-n} d((y_{n+1},y_{n+2},\ldots),(x_{n+1},x_{n+2},\ldots)) < 3^{-n}\varepsilon < \varepsilon \end{aligned}$$

and therefore

$$d(\phi(\bar{y}),\phi(\tilde{x})) < 3^{-K}.$$

It follows from the definition of ϕ that $\phi(\bar{y}) = \phi(\tilde{x})$. Thus

$$\begin{aligned} d_p^{\phi_n}(\bar{y},\tilde{x}) &= \max_{k=0,\ldots,p-1}\left\{\sum_{j=1}^{\infty} 3^{-j}|(\phi_n^k(\bar{y}))_j - (\phi_n^k(\tilde{x}))_j|\right\}\\ &= \max_{k=0,\ldots,p-1}\left\{\sum_{j=1}^{n} 3^{-j}|(\phi_n^k(\bar{y}))_j - (\phi_n^k(\tilde{x}))_j| + \sum_{j=n+1}^{\infty} 3^{-j}|(\phi_n^k(\bar{y}))_j - (\phi_n^k(\tilde{x}))_j|\right\}\\ &= \max_{k=0,\ldots,p-1}\left\{\sum_{j=1}^{n} 3^{-j}|(\phi^k(\bar{y}))_j - (\phi^k(\tilde{x}))_j| + \sum_{i=n+1}^{\infty} 3^{-j}|(\phi_n^k(\bar{y}))_j - (\phi_n^k(\tilde{x}))_j|\right\}\\ &= \max_{k=0,\ldots,p-1}\left\{3^{-n}\sum_{j=1}^{\infty} 3^{-j}|(\psi_a^k(\bar{y}^n))_j - (\psi_a^k(\tilde{x}^n))_j|\right\}\\ &= 3^{-n} d_p^{\psi_a}((y_{n+1},y_{n+2},\ldots),(x_{n+1},x_{n+2},\ldots)) < 3^{-n}\varepsilon. \end{aligned}$$

This fact proves $\tilde{B}$ is a $(p,\phi_n,3^{-n}\varepsilon)$-spanning set. Hence

$$\mathrm{span}(p,\phi_n,3^{-n}\varepsilon) \le \mathrm{span}(p,\psi_a,\varepsilon)\cdot\mathrm{span}(p,\phi,\varepsilon)$$

and thus

$$\mathrm{span}(\phi_n,3^{-n}\varepsilon) \le \mathrm{span}(\psi_a,\varepsilon) + \mathrm{span}(\phi,\varepsilon).$$

Therefore

$$\begin{aligned} \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\phi_n,3^{-n}\varepsilon)}{|\log 3^{-n}\varepsilon|} &= \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\phi_n,3^{-n}\varepsilon)}{|\log 3^{-n} + \log\varepsilon|}\\ &\le \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\psi_a,\varepsilon)}{|\log\varepsilon|} + \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\phi,\varepsilon)}{|\log\varepsilon|}\\ &= \limsup_{\varepsilon\to 0}\frac{\mathrm{span}(\psi_a,\varepsilon)}{|\log\varepsilon|}, \end{aligned}$$

which proves that $\overline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}},d,\phi_n) \le \overline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}},d,\psi_a)$. Analogously we can prove that $\underline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}},d,\psi_a) = \underline{\mathrm{mdim}}_{\mathrm{M}}(X_m^{\mathbb{N}},d,\phi_n)$. These facts proves the theorem. $\square$

Remark 5.4. If $p = \frac{j \log 2}{(j+1)\log 3}$ for some $j \in \mathbb{N}$, it follows from Proposition 5.1 and Theorem 5.3 that C_p is dense in $C^0(X_m^{\mathbb{N}})$.

Block, in [2], studied the continuity of the topological entropy map on the set consisting of continuous maps on the Cantor set, the interval and the circle. On the continuity of the metric mean dimension on the set consisting of continuous maps on the product space $X_m^{\mathbb{K}}$ (in particular on the Cantor set), we have from (5.1), Proposition 5.1 and Theorem 5.3 that:

Theorem 5.5. *If* $m \geq 2$*, then* $\mathrm{mdim}_{\mathrm{M}} : C^0(X_m^{\mathbb{K}}) \to \mathbb{R}$ *is not continuous anywhere. In particular,* $\mathrm{mdim}_{\mathrm{M}} : C^0(C) \to \mathbb{R}$ *is not continuous anywhere.*

It is well-known that any perfect, compact, metrizable, zero-dimensional space is homeomorphic to the middle third Cantor set (see [7], Chapter 6). Hence, suppose that X is a perfect, compact, metrizable, zero-dimensional space and let $\psi : X \to \boldsymbol{C}$ be an homeomorphism. Consider the metric on X given by

$$d_\psi(x, y) = d(\psi(x), \psi(y)) \quad \text{for } x, y \in X,$$

where d is the metric given in (5.2). Note that if ρ is other metric on X which induces the same topology that d_ψ on X, then $\hat{\rho}(\phi, \varphi) = \max_{x \in X} \rho(\phi(x), \varphi(x))$, for any $\phi, \varphi \in C^0(X)$, induces the same topology on $C^0(X)$ that the metric $\hat{d}_\psi(\phi, \varphi) = \max_{x \in X} d_\psi(\phi(x), \varphi(x))$. Therefore, the continuity of $\mathrm{mdim}_{\mathrm{M}} : C^0(X) \to \mathbb{R} \cup \{\infty\}$ does not depend on equivalent metrics on X. It follows from Theorem 5.5 that:

Corollary 5.6. *Suppose that* X *is a perfect, compact, metrizable, zero-dimensional space endowed with the metric* d_ψ*. The map* $\mathrm{mdim}_{\mathrm{M}} : C^0(X, d_\psi) \to \mathbb{R}$ *is not continuous anywhere. Therefore, for any perfect, compact, metric, zero-dimensional space* (X, d)*, the map* $\mathrm{mdim}_{\mathrm{M}} : C^0(X, d_\psi) \to \mathbb{R}$ *is not continuous anywhere.*

Next, we will consider the map $\mathrm{mdim} : C^0(X) \to \mathbb{R} \cup \{\infty\}$. Note if X is a finite set, then $\dim(X^{\mathbb{K}}) = 0$. Therefore, $\mathrm{mdim} : C^0(X^{\mathbb{K}}) \to \mathbb{R}$ is a constant map. More generally, if $(X_i)_{i \in \mathcal{J}}$ is a family of compact Hausdorff spaces with $\dim(X_i) = 0$ for each $i \in \mathcal{J}$, then $\dim(\prod_{i \in \mathcal{J}} X_i) = 0$. Hence $\mathrm{mdim} : C^0(\prod_{i \in \mathcal{J}} X_i) \to \mathbb{R}$ is a constant map. We will suppose that X is an n ($n \geq 1$) dimensional compact metric space, with metric d. We endow $X^{\mathbb{K}}$ with the product topology, which is obtained from any metric equivalent to the metric

$$\begin{aligned}&\tilde{d}((x_1, x_2, \dots), (y_1, y_2, \dots)) \\ &\quad = \sum_{n=1}^{\infty} 3^{-n} d(x_n, y_n) \quad \text{for any } (x_1, x_2, \dots), (y_1, y_2, \dots) \in X^{\mathbb{N}},\end{aligned}$$

for $\mathbb{K} = \mathbb{N}$ and

$$\tilde{d}((\ldots, x_{-1}, x_0, x_1, \ldots), (\ldots, y_{-1}, y_0, y_1, \ldots)) = \sum_{n \in \mathbb{Z}} 3^{-|n|} d(x_n, y_n),$$

for any $(\ldots, x_{-1}, x_0, x_1, \ldots), (\ldots, y_{-1}, y_0, y_1, \ldots) \in X^{\mathbb{Z}}$, for $\mathbb{K} = \mathbb{Z}$.

Theorem 5.7. *Take $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$ and X any finite dimensional compact metric space.*

i. The set consisting of continuous maps on $X^{\mathbb{K}}$ with zero mean dimension is dense in $C^0(X^{\mathbb{K}})$.
ii. If there exists $\psi_a \in C^0(X^{\mathbb{K}})$ with mean dimension equal to a, then the set consisting of continuous maps with mean dimension equal to a is dense in $C^0(X^{\mathbb{K}})$.
iii. $\mathrm{mdim} : C^0(X^{\mathbb{K}}) \to \mathbb{R} \cup \{\infty\}$ *is constant or is not continuous anywhere.*

Proof. We consider $\mathbb{K} = \mathbb{N}$. We will fix a continuous map $\phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$, given by $\phi(x_1, x_2, \ldots) = (y_1(\bar{x}), y_2(\bar{x}), \ldots)$, for any $\bar{x} = (x_1, x_2, \ldots) \in X^{\mathbb{N}}$. Consider the sequence of continuous maps on $X^{\mathbb{N}}$, $(\phi_n)_{n \in \mathbb{N}}$, defined by

$$\phi_n(\bar{x}) = (y_1(\bar{x}), y_2(\bar{x}), \ldots, y_n(\bar{x}), x_0, x_0, \ldots) \text{ for any } n \in \mathbb{N} \text{ and some } x_0 \in X.$$

Note that $\Omega(\phi_n) \subseteq \underbrace{X \times \cdots \times X}_{n\text{-times}} \times \{x_0\} \times \cdots$, and then $\Omega(\phi_n)$ is a finite dimensional space. Hence $\mathrm{mdim}(\phi_n, X^{\mathbb{N}}) = \mathrm{mdim}(\phi_n|_{\Omega(\phi_n)}, \Omega(\phi_n)) = 0$, since any continuous map on a finite dimensional space has mean dimension equal to zero. Note that ϕ_n converges uniformly to ϕ as $n \to \infty$. This fact proves i.

We prove ii. Suppose there exists $\psi_a \in C^0(X^{\mathbb{N}})$, which is given by

$$\psi_a(\bar{x}) = (z_1(\bar{x}), z_2(\bar{x}), \ldots) \quad \text{for any } \bar{x} = (x_1, x_2, \ldots) \in X^{\mathbb{N}},$$

and $\mathrm{mdim}(\phi_n, X^{\mathbb{N}}) = a > 0$. Fix $\phi \in C^0(X^{\mathbb{N}})$, which, without loss of generality, we can suppose that

$$\phi(\bar{x}) = (y_1(\bar{x}), y_2(\bar{x}), \ldots, y_K(\bar{x}), z_0, z_0, \ldots) \text{ for some } K \in \mathbb{N} \text{ and some } z_0 \in X,$$

for any $\bar{x} = (x_1, x_2, \ldots) \in X^{\mathbb{N}}$. For each $n \geq K + 1$, if $\bar{x} = (x_1, x_2, \ldots)$, set

$$\bar{x}^n = (x_{n+1}, x_{n+2}, \ldots) \quad \text{and} \quad \bar{x}_n = (x_1, x_2, \ldots, x_n, z_0, z_0, \ldots).$$

Consider the sequence of continuous maps on $X^{\mathbb{N}}$, $(\phi_n)_{n \geq K+1}$, where

$$\phi_n(\bar{x}) = (y_1(\bar{x}_n), y_2(\bar{x}_n), \ldots, y_K(\bar{x}_n), \underbrace{z_0, \ldots, z_0}_{(n-K)\text{-times}}, z_1(\bar{x}^n), z_2(\bar{x}^n), \ldots) \text{ for } n \geq K + 1 \text{ and } \bar{x} \in X^{\mathbb{N}}.$$

We have ϕ_n converges uniformly to ϕ as $n \to \infty$. On the other hand, note that

$$\Omega(\phi_n) \subseteq \underbrace{X \times \cdots \times X}_{K\text{-times}} \times \underbrace{\{z_0\} \times \cdots \times \{z_0\}}_{(n-K)\text{-times}} \times X \times X \times \cdots := Z.$$

Hence, we can consider the restriction $\phi_n|_Z : Z \to Z$ in order to find the mean dimension of ϕ_n. Define $\mathcal{I} : X^{\mathbb{N}} \to X^{\mathbb{N}} \times X^{\mathbb{N}}$, defined by $\mathcal{I}(\bar{x}) = (\bar{x}_n, \bar{x}^n)$, and $\Phi_n : X^{\mathbb{N}} \times X^{\mathbb{N}} \to X^{\mathbb{N}} \times X^{\mathbb{N}}$, defined by $\Phi_n(\bar{x}, \bar{y}) = (\phi(\bar{x}_n), \psi_a(\bar{y}))$. We have

$$\Phi_n(\mathcal{I}(\bar{x})) = \Phi_n(\bar{x}_n, \bar{x}^n) = (\phi(\bar{x}_n), \psi_a(\bar{x}^n)) = \mathcal{I}(\phi_n(\bar{x})),$$

Hence,

$$\begin{aligned} &\operatorname{mdim}(X^{\mathbb{N}}, \phi_n) \leq \operatorname{mdim}(X^{\mathbb{N}} \times X^{\mathbb{N}}, \Phi_n) \leq \operatorname{mdim}(X^{\mathbb{N}}, \phi) + \operatorname{mdim}(X^{\mathbb{N}}, \psi_a) \\ &\quad = \operatorname{mdim}(X^{\mathbb{N}}, \psi_a). \end{aligned}$$

On the other hand, we can refine each open cover of Z to one of the form

$$\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_K \times \underbrace{\{z_0\} \times \cdots \times \{z_0\}}_{(n-K)\text{-times}} \times \mathcal{A}_{n+1} \times \mathcal{A}_{n+2} \times \cdots ,$$

where $\mathcal{A}_i$ is an open cover of X and, for some J, $\mathcal{A}_i = X$ for all $i \geq J$. Set

$$\begin{aligned} \mathcal{B} &= \mathcal{A}_{n+1} \times \mathcal{A}_{n+2} \times \cdots \\ \tilde{\mathcal{B}} &= X \times \cdots \times X \times \{z_0\} \times \cdots \times \{z_0\} \times \mathcal{B} \\ \mathcal{B}_0^m(\psi_a) &= \mathcal{B} \vee (\psi_a^{-1}(\mathcal{B})) \vee \cdots \vee (\psi_a^{-m}(\mathcal{B})) \\ \mathcal{A}_0^m(\phi_n) &= \mathcal{A} \vee (\phi_n^{-1}(\mathcal{A})) \vee \cdots \vee (\phi_n^{-m}(\mathcal{A})) \\ \tilde{\mathcal{B}}_0^m(\phi_n) &= \tilde{\mathcal{B}} \vee (\phi_n^{-1}(\tilde{\mathcal{B}})) \vee \cdots \vee (\phi_n^{-m}(\tilde{\mathcal{B}})). \end{aligned}$$

Let $\pi : X^{\mathbb{N}} \to X^{\mathbb{N}}$, given by $\pi(x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_{n+1}, x_{n+2}, \ldots)$. Note that

$$\pi(\tilde{\mathcal{B}}_0^m(\phi_n)) \succ \mathcal{B}_0^m(\psi_a) \quad \text{and} \quad \mathcal{A}_0^m(\phi_n) \succ \tilde{\mathcal{B}}_0^m(\phi_n).$$

Hence

$$\mathcal{D}(\mathcal{B}_0^m(\psi_a)) \leq \mathcal{D}(\pi(\tilde{\mathcal{B}}_0^m(\phi_n))) \leq \mathcal{D}(\tilde{\mathcal{B}}_0^m(\phi_n)) \leq \mathcal{D}(\mathcal{A}_0^m(\phi_n)) \quad \text{for each } m \in \mathbb{N}.$$

Therefore

$$\lim_{m\to\infty} \frac{\mathcal{D}(\mathcal{B}_0^m(\psi_a))}{m+1} \leq \lim_{m\to\infty} \frac{\mathcal{D}(\mathcal{A}_0^m(\phi_n))}{m+1},$$

which proves that

$$\operatorname{mdim}(\phi_n, X^{\mathbb{N}}) \geq \operatorname{mdim}(\psi_a, X^{\mathbb{N}}) \quad \text{for any } n \geq K+1.$$

Note that iii follows from i and ii. □

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References

[1] Artin, M., Mazur, B.: On periodic points. Ann. Math., 82–99 (1965)

[2] Block, L.: Noncontinuity of topological entropy of maps of the Cantor set and of the interval. Proc. Am. Math. Soc. **50**(1), 388–393 (1975)

[3] Bobok, J., Zindulka, O.: Topological entropy on zero-dimensional spaces. Fundam. Math. **162**(3), 233–249 (1999)

[4] Carvalho, M., Rodrigues, F.B., Varandas, P.: Generic homeomorphisms have full metric mean dimension. Ergodic Theory Dyn. Syst., 1–25 (2019)

[5] De Melo, W., Van Sebastian, S.: One-Dimensional Dynamics, vol. 25. Springer, Berlin (2012)

[6] do Carmo., M.P.: Geometria riemanniana. Instituto de Matemática Pura e Aplicada (2008)

[7] Engelking, R.: "General Topology. Heldermann, Berlin." MR1039321 (91c: 54001): 529 (1989)

[8] Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications. Wiley, Hoboken (2004)

[9] Gromov, M.: Topological invariants of dynamical systems and spaces of holomorphic maps: I. Math. Phys. Anal. Geom. **2**(4), 323–415 (1999)

[10] Gutman, Y.: Embedding topological dynamical systems with periodic points in cubical shifts. Ergodic Theory Dyn. Syst. **37**(2), 512–538 (2017)

[11] Gutman, Y., Tsukamoto, M.: Embedding minimal dynamical systems into Hilbert cubes. Invent. Math. **221**(1), 113–166 (2020)

[12] Hurley, M.: On proofs of the C^0 general density theorem. Proc. Am. Math. Soc. **124**(4), 1305–1309 (1996)

[13] Jin, L., Yixiao, Q.: Mean dimension of product spaces: a fundamental formula. arXiv preprint arXiv:2102.10358 (2021)

[14] Lindenstrauss, E.: Mean dimension, small entropy factors and an embedding theorem. Publications Mathématiques de l'Institut des Hautes Études Scientifiques **89**(1), 227–262 (1999)

[15] Lindenstrauss, E., Weiss, B.: Mean topological dimension. Israel J. Math. **115**(1), 1–24 (2000)

[16] Lindenstrauss, E., Tsukamoto, M.: From rate distortion theory to metric mean dimension: variational principle. IEEE Trans. Inf. Theory **64**(5), 3590–3609 (2018)

[17] Lindenstrauss, E., Tsukamoto, M.: Mean dimension and an embedding problem: an example. Israel J. Math. **199**(2), 573–584 (2014)

[18] Misiurewicz, M.: Horseshoes for Continuous Mappings of an Interval. Dynamical Systems, pp. 125–135. Springer, Berlin (2010)

[19] Newhouse, S.E.: Continuity properties of entropy. Ann. Math. **129**(1), 215–235 (1989)

[20] Rodrigues, F.B., Jeovanny, M.A.: Mean dimension and metric mean dimension for non-autonomous dynamical systems. J. Dyn. Control Syst. 1–27 (2021)

[21] Tsukamoto, M.: Mean dimension of full shifts. Israel J. Math. **230**(1), 183–193 (2019)

[22] Velozo, A., Renato, V.: Rate distortion theory, metric mean dimension and measure theoretic entropy. arXiv preprint arXiv:1707.05762 (2017)

[23] Wei, C., Wen, S., Wen, Z.: Remarks on dimensions of Cartesian product sets. Fractals **24**(03), 1650031 (2016)

[24] Yano, K.: A remark on the topological entropy of homeomorphisms. Invent. Math. **59**(3), 215–220 (1980)

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