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Mean Dimension and Metric Mean Dimension for Non-autonomous Dynamical Systems

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Abstract

In this paper we extend the definitions of mean dimension and metric mean dimension for non-autonomous dynamical systems. We show some properties of this extension and furthermore some applications to the mean dimension and metric mean dimension of single continuous maps.

Keywords Non-autonomous dynamical systems · Mean dimension · Metric mean dimension · Topological entropy

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1 Introduction

In the late 1990s, M. Gromov in [2] introduced the notion of mean dimension for a topological dynamical system (X, ϕ) (X is a compact topological space and ϕ is a continuous map on X), which is, as well as the topological entropy, an invariant under conjugacy. In [11], Lindenstrauss and Weiss showed that the mean dimension is zero if the topological dimension of X is finite. They gave some examples where the mean dimension is positive. For instance, they proved that the mean dimension of $\big(([0,1]^m)^{\mathbb{Z}}, \sigma\big)$, where σ is the two-sided full shift map on $([0,1]^m)^{\mathbb{Z}}$, which has infinite topological entropy, is equals to m and that any non-trivial factor of $\big(([0,1]^m)^{\mathbb{Z}}, \sigma\big)$ has positive mean dimension.

Given a dynamical system (X, ϕ), an interesting question related to such a system is the following: under what conditions is it possible to imbed such a system in the shift

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space $\left(\left([0,1]^{\mathbb{N}}\right)^{\mathbb{Z}},\sigma\right)$? That is, what properties the system must have to guarantee the existence of a continuous map $i : X \to \left([0,1]^{\mathbb{N}}\right)^{\mathbb{Z}}$ satisfying $\sigma \circ i = i \circ \varphi$? In [11] the authors proved that a necessary condition for an invertible system (X,ϕ) to be embedded in $\left(([0,1]^m)^{\mathbb{Z}},\sigma\right)$ is that $\mathrm{mdim}(X,\phi) \leq m$, where $\mathrm{mdim}(X,\phi)$ denotes the mean dimension of the system (X,ϕ). In [12] it was proved that if (X,ϕ) is an invertible system which is an extension of a minimal system, and K is a convex set with non-empty interior such that $\mathrm{mdim}(X,\phi) < \dim K/36$, then (X,ϕ) can be embedded in the shift space $\left(K^{\mathbb{Z}},\sigma\right)$. In particular, if $\mathrm{mdim}(X,\phi) < m/36$, then (X,ϕ) can be embedded in $\left(([0,1]^m)^{\mathbb{Z}},\sigma\right)$. More recently, Gutman and Tsukamoto [4] showed that, that if (X,ϕ) is a minimal system with $\mathrm{mdim}(X,\phi) < N/2$ then we can embed it in $\left(([0,1]^N)^{\mathbb{Z}},\sigma\right)$. In [13, Theorem 1.3], Lindenstrauss and Tsukamoto constructed a minimal system with mean dimension equal to $N/2$ which cannot be embedded into $\left(([0,1]^N)^{\mathbb{Z}},\sigma\right)$, showing that the constant $N/2$ obtained in [4] is optimal.

The notion of metric mean dimension for a dynamical system $\phi : (X,d) \to (X,d)$ was introduced in [11], where (X,d) is a compact metric space with metric d and ϕ is a continuous map. It refines the topological entropy for systems with infinite entropy, which, in the case of a manifold of dimension greater than one, form a residual subset of the set consisting of homeomorphisms defined on the manifold (see [18]). In fact, every system with finite topological entropy has metric mean dimension equals to zero and for any metric d' equivalent to d on X one has $\mathrm{mdim}(X,\phi) \leq \mathrm{mdim}_M(X,\phi,d')$, where $\mathrm{mdim}_M(X,\phi,d')$ denotes the metric mean dimension of (X,ϕ) with respect to d' (see [10, 11]). The metric mean dimension depends on the metric d, therefore it is not a topological invariant. However, for a metrizable topological space X, $\mathrm{mdim}_M(X,\phi) = \inf_{d'} \mathrm{mdim}_M(X,\phi,d')$ is invariant under topological conjugacy, where the infimum is taken over all the metrics on X which induce the topology on X. In [10], Theorem 4.3, the author proved that if (X,ϕ) is an extension of a minimal system, then there exists a metric d' on X, equivalent to d, such that $\mathrm{mdim}(X,\phi) = \mathrm{mdim}_M(X,\phi,d')$.

B. Kloeckner [7] studied the dynamical system $\left(\mathcal{P}\left(\mathbb{S}^1\right),\Phi_{d\sharp}\right)$, where $\mathcal{P}\left(\mathbb{S}^1\right)$ is the space of probability measures on the circle $\mathbb{S}^1$ and $\Phi_{d\sharp}$ is the push-forward map induced by a d-expanding map $\Phi_d : \mathbb{S}^1 \to \mathbb{S}^1$. The author shows if we take the Wasserstein metric with cost function $|\cdot|^p$ ($p \in [1,\infty)$) on $\mathcal{P}\left(\mathbb{S}^1\right)$, denoted by $\mathcal{W}_p$, then $\mathrm{mdim}_M\left(\mathcal{P}\left(\mathbb{S}^1\right),\Phi_{d\sharp},\mathcal{W}_p\right) \geq p(d-1)$. H. Lee (in [9]) introduced the mean dimension for continuous actions of countable sofic groups on compact metrizable spaces and proved that, in this setting, the mean dimension is an important tool for distinguishing continuous actions of countable sofic groups with infinite entropy.

A *non-autonomous dynamical system* (or a sequential dynamical system) is a sequence $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ of continuous maps $f_n : X_n \to X_{n+1}$, where X_n is a compact topological space for every $n \in \mathbb{N}$. In the last two decades, several authors have tried to extend some results that are valid for autonomous systems for the non-autonomous case. Kolyada and Snoda in [8] introduced the notion of topological entropy for this setting and proved that, just as in the case of autonomous systems, it is an invariant under equiconjugacy and furthermore that it is concentrated in the non-wandering set of the dynamics (see [8] and [15]). In a more recent work, Freitas et al. [1] have analyzed the existence of Extreme Value Laws in this setting. In [16] Stadlbauer guarantees, under appropriate conditions, the existence of a spectral gap for transference operators associated with sequential systems.

As we said above, the set consisting of continuous maps with infinite topological entropy is residual. On the other hand, it is easy to build non-autonomous dynamical systems with infinite topological entropy (take ϕ a continuous map with positive topological entropy, then

$\left(\phi, \phi^2, \phi^{2^2}, \phi^{2^3}, \dots\right)$ is a non-autonomous dynamical systems with infinite topological entropy). This is the main reason to extend the concepts of mean dimension and metric mean dimension to non-autonomous systems, since these become a tool to classify non-autonomous dynamical systems with infinite topological entropy (see Theorem 6.1).

In the next two sections we will extend the mean dimension and the metric mean dimension for a non-autonomous dynamical system $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$, which will be denoted by $\mathrm{mdim}(X, \boldsymbol{f})$. Furthermore, we will prove some properties which are valid for the entropy of non-autonomous dynamical systems (see [8] and [15]). An application of these properties is that, for any continuous maps ϕ and ψ on X, the compositions $\phi \circ \psi$ and $\psi \circ \phi$ have the same mean dimension (see Corollary 2.7). Furthermore, Remark 4.2 proves the inequality $\mathrm{mdim}_M(X, \phi^p, d) \leq p\, \mathrm{mdim}_M(X, \phi, d)$ can be strict. Proposition 3.5 proves if $X = [0, 1]$ or $\mathbb{S}^1$, then for each $a \in [0, 1]$, there exists a continuous map ϕ_a on X with metric mean dimension equals to a. In Theorem 4.6 we show that, as the topological entropy, the metric mean dimension is concentrated in the non-wandering set of the dynamics.

In Section 5 we will discuss some upper bounds for the metric mean dimension of both autonomous and non-autonomous dynamical systems.

As we said above, the metric mean dimension for single continuous maps, and consequently for non-autonomous dynamical systems, depends on the metric d. In Section 6 we will discuss some properties related to the invariance of the metric mean dimension under topological equiconjugacy.

In the last section we will present some results related to the continuity of the metric mean dimension.

Some ideas given to proof the results that are well-known for the autonomous case work or can be adapted for the non-autonomous case. We will present these proofs for the sake of comprehensiveness.

2 Mean Dimension for Non-autonomous Dynamical Systems

Let X be a compact metric space. In this section we will suppose that $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ is a non-autonomous dynamical system, where $f_n : X \to X$ is a continuous map for all $n \geq 1$. We write $(X, \boldsymbol{f}, d)$ to denote a non-autonomous dynamical system $\boldsymbol{f}$ on X endowed with the metric d. For $n, k \in \mathbb{N}$ define

$$f_n^{(0)} := I_X := \text{the identity on } X \qquad \text{and} \qquad f_n^{(k)}(x) := f_{n+k-1} \circ \cdots \circ f_n(x) \quad \text{for } k \geq 1.$$

Set

$$\mathcal{C}(X) = \left\{ (f_n)_{n=1}^{\infty} : f_n : X \to X \text{ is a continuous map} \right\}.$$

Given α an open cover of X define

$$\alpha_0^{n-1} = \alpha \vee f_1^{-1}(\alpha) \vee \left(f_1^{(2)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_1^{(n-1)}\right)^{-1}(\alpha)$$

and set

$$\mathrm{ord}(\alpha) = \sup_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1 \qquad \text{and} \qquad \mathcal{D}(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}\,(\beta),$$

where 1_U is the indicator function and $\beta \succ \alpha$ means that β is an open cover of X finner than α.

Definition 2.1 The *mean dimension* of $\boldsymbol{f} \in \mathcal{C}(X)$ is defined to be

$$\mathrm{mdim}(X, \boldsymbol{f}) = \sup_{\alpha} \lim_{n\to\infty} \frac{\mathcal{D}\left(\alpha_0^{n-1}\right)}{n}.$$

By Corollary 2.5 of [11] we have that $\mathcal{D}(\alpha \vee \beta) \leq \mathcal{D}(\alpha) + \mathcal{D}(\beta)$, for any open covers α and β. It follows that the limit that defines the mean dimension is well defined.

Remark 2.2 We present a list of some important properties about the mean dimension for both autonomous and non-autonomous dynamical systems:

(1) For a non-autonomous dynamical system given by the iterates of a single continuous map $f : X \to X$, i.e., $\boldsymbol{f} = (f)_{n=1}^{\infty}$, the definition of mean dimension coincides with the one presented in [11], that is, $\mathrm{mdim}\left(X, (f)_{n=1}^{\infty}\right) = \mathrm{mdim}(X, f)$.

(2) Recall that for a topological space X, the *topological dimension* is defined as

$$\dim(X) = \sup_{\alpha} \mathcal{D}(\alpha)$$

where α runs the open covers of X. If $\dim(X) < \infty$, then $\mathcal{D}\left(\alpha_0^{n-1}\right) \leq \dim(X)$ for all $n \in \mathbb{N}$ and therefore $\mathrm{mdim}(X, \boldsymbol{f}) = 0$ for any $\boldsymbol{f} \in \mathcal{C}(X)$.

(3) In [11], Proposition 3.1, is proved that $\mathrm{mdim}\left(X^{\mathbb{Z}}, \sigma\right) \leq \dim(X)$, where σ is the shift on $X^{\mathbb{Z}}$. Analogously we can prove $\mathrm{mdim}\left(X^{\mathbb{N}}, \sigma\right) \leq \dim(X)$.

(4) If $X = [0, 1]^m$, then $\mathrm{mdim}\left(X^{\mathbb{Z}}, \sigma\right) = m$ (see [11], Proposition 3.3).

(5) It is clear that if $Y \subseteq X$ is an invariant subset by a continuous map $\phi : X \to X$, then $\mathrm{mdim}(Y, \phi) \leq \mathrm{mdim}(X, \phi)$. We can define the mean dimension for any $Y \subseteq X$ as follows: let α be an open cover of X and consider $\alpha|_Y = \{U \cap Y : U \in \alpha\}$, the open cover of Y given by the restriction of α to Y. Then define

$$\mathrm{mdim}(Y, \boldsymbol{f}|_Y) = \sup_{\alpha} \lim_{n\to\infty} \frac{\mathcal{D}\left((\alpha|_Y)_0^{n-1}\right)}{n}.$$

It is clear that $\mathrm{mdim}(Y, \boldsymbol{f}|_Y) \leq \mathrm{mdim}(X, \boldsymbol{f})$.

(6) A necessary condition for an invertible dynamical system $\phi : X \to X$ to be imbeddable in $\left(([0, 1]^m)^{\mathbb{Z}}, \sigma\right)$ is that $\mathrm{mdim}(X, \phi) \leq m$ (see [11], Corollary 3.4).

(7) Any non-trivial factor of $\left([0, 1]^{\mathbb{Z}}, \sigma\right)$ has positive mean dimension (see [11], Theorem 3.6).

We will show some properties of the mean dimension which are valid for the topological entropy. Denote by $h_{top}(\boldsymbol{f})$ the topological entropy of $\boldsymbol{f}$ (see [8, 15]).

Definition 2.3 For any $p \geq 1$, set

$$\boldsymbol{f}^{(p)} = \left\{f_1^{(p)}, f_{p+1}^{(p)}, f_{2p+1}^{(p)}, \dots\right\} = \{f_p \circ \cdots \circ f_1, f_{2p} \circ \cdots \circ f_{p+1}, f_{3p} \circ \cdots \circ f_{2p+1}, \dots\}.$$

It is well-known that $h_{top}(\phi^p) = p\, h_{top}(\phi)$ for any $p \geq 1$, where ϕ is any continuous map. For non-autonomous dynamical systems we have

$$h_{top}\left(\boldsymbol{f}^{(p)}\right) \leq p\, h_{top}(\boldsymbol{f}) \quad \text{for any } p \geq 1$$

(see [8], Lemma 4.2). In general, the equality $h_{top}\left(\boldsymbol{f}^{(p)}\right) = p\, h_{top}(\boldsymbol{f})$ is not valid, as we can see in the next example, which was given by Kolyada and Snoha in [8].

Example 2.4 Take $\psi : [0, 1] \to [0, 1]$ defined by $\psi(x) = 1 - |2x - 1|$ for any $x \in [0, 1]$. Consider $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$, where

$$f_n(x) = \begin{cases} \psi^{(n+1)/2}(x), & \text{if } n \text{ is odd,} \\ x/2^{n/2}, & \text{if } n \text{ is even,} \end{cases}$$

for any $n \in \mathbb{N}$. Then $h_{top}\left(\boldsymbol{f}^{(2)}\right) = 0$ and $h_{top}(\boldsymbol{f}) \geq \frac{\log 2}{2}$.

The equality $h_{top}\left(\boldsymbol{f}^{(p)}\right) = p\, h_{top}(\boldsymbol{f})$ is valid if the sequence $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ is equicontinuous (see [8], Lemma 4.4). On the other hand, the equality always holds for the mean dimension.

Proposition 2.5 *For any $\boldsymbol{f} = (f_n)_{n=1}^{\infty} \in \mathcal{C}(X)$ and $p \in \mathbb{N}$ we have*

$$\operatorname{mdim}\left(X, \boldsymbol{f}^{(p)}\right) = p \operatorname{mdim}(X, \boldsymbol{f}).$$

Proof Let α be an open cover of X. Note that, for $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \frac{\mathcal{D}\left(\alpha \vee \left(f_1^{(p)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_1^{((k-1)p)}\right)^{-1}(\alpha)\right)}{k} \leq \lim_{k\to\infty} \frac{\mathcal{D}\left(\alpha_0^{(k-1)p}\right)}{k} \leq p \lim_{k\to\infty} \frac{\mathcal{D}\left(\alpha_0^{(kp-1)}\right)}{kp},$$

which implies that $\operatorname{mdim}\left(X, \boldsymbol{f}^{(p)}\right) \leq p \operatorname{mdim}(X, \boldsymbol{f})$. For the converse, note that

$$\alpha_0^{kp-1} = \alpha_0^{p-1} \vee \left(f_1^{(p)}\right)^{-1}\left(\alpha_0^{p-1}\right) \vee \left(f_1^{(2p)}\right)^{-1}\left(\alpha_0^{p-1}\right) \vee \cdots \vee \left(f_1^{((k-1)p)}\right)^{-1}\left(\alpha_0^{p-1}\right),$$

and therefore

$$\operatorname{mdim}(X, \boldsymbol{f}) = \sup_{\alpha} \lim_{k\to\infty} \frac{\mathcal{D}\left(\alpha_0^{(k-1)p}\right)}{kp} \leq \frac{\operatorname{mdim}\left(X, \boldsymbol{f}^{(p)}\right)}{p},$$

which proves the proposition. □

In [8], Lemma 4.5, Kolyada and Snoha proved that

$$h_{top}\left(\sigma^i(\boldsymbol{f})\right) \leq h_{top}\left(\sigma^j(\boldsymbol{f})\right) \quad \text{for any } i \leq j,$$

where σ is the left shift $\sigma((f_n)_{n=1}^{\infty}) = (f_{n+1})_{n=1}^{\infty}$. Furthermore, in [15], Corollary 5.6, the author showed that if each f_n is an homeomorphism then the equality holds, that is, the topological entropy for non-autonomous dynamical systems is independent on the first maps on a sequence of homeomorphisms $\boldsymbol{f} = (f_n)_{n\in\mathbb{Z}}$. Next proposition shows that these properties also hold for the mean dimension.

Proposition 2.6 *Let i, j be two positive integers with $i \leq j$. Then*

$$\operatorname{mdim}\left(X, \sigma^i(\boldsymbol{f})\right) \leq \operatorname{mdim}\left(X, \sigma^j(\boldsymbol{f})\right).$$

If each f_n is a homeomorphism then the equality holds.

Proof It is enough to prove the proposition for $i = 0$ and $j = 1$. For any open cover α of X we have

$$\begin{aligned}\mathcal{D}\left(\alpha_0^{n-1}\right) &\leq \mathcal{D}(\alpha) + \mathcal{D}\left(f_1^{-1}(\alpha \vee (f_2)^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_2^{(n-2)}\right)^{-1}(\alpha))\right)\\ &= \mathcal{D}(\alpha) + \mathcal{D}\left(\alpha \vee (f_2)^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_2^{(n-2)}\right)^{-1}(\alpha)\right).\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{\mathcal{D}\left(\alpha_0^{n-1}\right)}{n} &\leq \lim_{n\to\infty}\frac{\mathcal{D}(\alpha)}{n} + \lim_{n\to\infty}\frac{\mathcal{D}\left(\alpha \vee f_2^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_2^{(n-2)}\right)^{-1}(\alpha)\right)}{n}\\ &= \lim_{n\to\infty}\frac{n-1}{n}\frac{\mathcal{D}\left(\alpha \vee f_2^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \cdots \vee \left(f_2^{(n-2)}\right)^{-1}(\alpha)\right)}{n-1}\\ &\leq \operatorname{mdim}(X, \sigma(\boldsymbol{f})),\end{aligned}$$

and therefore $\operatorname{mdim}(X,\boldsymbol{f}) \leq \operatorname{mdim}(X, \sigma(\boldsymbol{f}))$.

Next, suppose that each f_n is a homeomorphism. Note that if β refines α then $\mathcal{D}(\beta) \geq \mathcal{D}(\alpha)$. Therefore, we have

$$\begin{aligned}\mathcal{D}\left(\alpha \vee (f_2)^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \dots\right) &= \mathcal{D}\left(f_1^{-1}\left(\alpha \vee (f_2)^{-1}(\alpha) \vee \left(f_2^{(2)}\right)^{-1}(\alpha) \vee \dots\right)\right)\\ &= \mathcal{D}\left((f_1)^{-1}(\alpha) \vee \left(f_1^{(2)}\right)^{-1}(\alpha) \vee \left(f_1^{(3)}\right)^{-1}(\alpha) \vee \dots\right)\\ &\leq \mathcal{D}\left(\alpha \vee (f_1)^{-1}(\alpha) \vee \left(f_1^{(2)}\right)^{-1}(\alpha) \vee \left(f_1^{(3)}\right)^{-1}(\alpha) \vee \dots\right).\end{aligned}$$

Hence $\operatorname{mdim}(X, \sigma(\boldsymbol{f})) \leq \operatorname{mdim}(X,\boldsymbol{f})$. □

If some f_n is not a homeomorphism, then the inequality above can be strict. In fact, take $f_n = f : X \to X$ for any $n \geq 2$, where f is any continuous map with positive mean dimension and $f_1 : X \to X$ a constant map. Then $\operatorname{mdim}(X,\boldsymbol{f}) = 0$ and $\operatorname{mdim}(X, \sigma(\boldsymbol{f})) = \operatorname{mdim}(X, f)$.

Next corollary follows from Propositions 2.5 and 2.6:

Corollary 2.7 *Let* $\boldsymbol{f} = (f, g, f, g, \dots)$ *and* $\boldsymbol{g} = (g, f, g, f, \dots)$*, where* $f, g : X \to X$ *are continuous maps. Then*

$$\operatorname{mdim}(X,\boldsymbol{f}) = \operatorname{mdim}(X, \boldsymbol{g}).$$

Therefore,

$$\operatorname{mdim}(X, f \circ g) = \operatorname{mdim}(X, g \circ f).$$

Proof It follows directly from Proposition 2.6 that $\operatorname{mdim}(X,\boldsymbol{f}) = \operatorname{mdim}(X, \boldsymbol{g})$. Now, by Proposition 2.5 we have

$$\begin{aligned}\operatorname{mdim}(X, f \circ g) &= \operatorname{mdim}\left(X,\boldsymbol{f}^{(2)}\right) = 2\operatorname{mdim}(X,\boldsymbol{f}) = 2\operatorname{mdim}(X, \boldsymbol{g})\\ &= \operatorname{mdim}\left(X, \boldsymbol{g}^{(2)}\right) = \operatorname{mdim}(X, g \circ f),\end{aligned}$$

which proves the corollary. □

It follows directly from Corollary 2.7 that if f and g are topologically conjugate continuous maps, then

$$\mathrm{mdim}(X, f) = \mathrm{mdim}(X, g),$$

since if ϕ is a topological conjugacy between f and g, that is, ϕ is a homeomorphism and $\phi \circ f = g \circ \phi$, then

$$\mathrm{mdim}(X, f) = \mathrm{mdim}\left(X, \phi^{-1} \circ \phi \circ f\right) = \mathrm{mdim}\left(X, \phi \circ f \circ \phi^{-1}\right) = \mathrm{mdim}(X, g).$$

For any $\boldsymbol{f} = (f_n)_{n=1}^{\infty} \in \mathcal{C}(X)$, the *asymptotic mean dimension* is defined by the limit

$$\mathrm{mdim}(X, \boldsymbol{f})^* = \lim_{n \to \infty} \mathrm{mdim}(X, \sigma^n(\boldsymbol{f})).$$

It follows from Proposition 2.6 that the asymptotic mean dimension always exists.

Theorem 2.8 *Let $\boldsymbol{f} = (f_n)_{n=1}^{\infty} \in \mathcal{C}(X)$. If $\boldsymbol{f}$ converges uniformly to a continuous map $f : X \to X$, then*

$$\mathrm{mdim}(X, \boldsymbol{f})^* \leq \mathrm{mdim}(X, f).$$

In particular, $\mathrm{mdim}(X, \boldsymbol{f}) \leq \mathrm{mdim}(X, f)$.

Proof Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of mutually different point converging to a point x_0. Define the map $F : \{x_n : n = 0, 1, \ldots\} \times X \to \{x_n : n = 0, 1, \ldots\} \times X$ by $F : (x, y) \mapsto (\phi(x), \psi(x, y))$, where

$$\phi(x_n) = \begin{cases} x_0, & \text{if } n = 0 \\ x_{n+1}, & \text{if } n > 0 \end{cases} \quad \text{and} \quad \psi(x_n, y) = \begin{cases} f(y), & \text{if } n = 0 \\ f_n(y), & \text{if } n > 0. \end{cases}$$

Note that the non-wandering set of F, $\Omega(F)$, is a subset of the fix fiber $x_0 \times X$. Since

$$\mathrm{mdim}(\{x_n : n = 0, 1, \ldots\} \times X, F) = \mathrm{mdim}(\Omega(F), F)$$

(by [3, Lemma 7.2]), we have that

$$\mathrm{mdim}(\{x_n : n = 0, 1, \ldots\} \times X, F) = \mathrm{mdim}(\{x_0\} \times X, F).$$

Therefore,

$$\mathrm{mdim}(\{x_m : m \geq k\} \times X, F) \leq \mathrm{mdim}(\{x_0\} \times X, F) = \mathrm{mdim}(\{x_n : n = 0, 1, \ldots\} \times X, F),$$

for all $k > 0$ (see Remark 2.2, item (3)). Next, note that by the definition of F we have that

$$\mathrm{mdim}(\{x_m : m \geq k\} \times X, F) = \mathrm{mdim}\left(X, \sigma^k(\boldsymbol{f})\right), \quad \text{for } k > 0,$$

and $\mathrm{mdim}(\{x_0\} \times X, F) = \mathrm{mdim}(X, f)$. Hence, $\mathrm{mdim}(X, \sigma^k(\boldsymbol{f})) \leq \mathrm{mdim}(X, f)$, for all k. □

Next example proves that the inequality above can be strict.

Example 2.9 Let $\phi : I^{\mathbb{N}} \to I^{\mathbb{N}}$ be a continuous map with positive mean dimension. For each $n \geq 1$, set $f_n : I^{\mathbb{N}} \times I^{\mathbb{N}} \to I^{\mathbb{N}} \times I^{\mathbb{N}}$ defined by

$$f_n\left((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}\right) = \left((\lambda_n x_i)_{i \in \mathbb{N}}, (x_i(\phi(y))_i)_{i \in \mathbb{N}}\right),$$

where $\lambda_n \to 1$ and $\lambda_n \cdots \lambda_1 \to 0$ as $n \to \infty$. Note that f_n converges uniformly on $I^{\mathbb{N}} \times I^{\mathbb{N}}$ to $f((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = ((x_i)_{i \in \mathbb{N}}, (x_i(\phi(y))_i)_{i \in \mathbb{N}})$ as $n \to \infty$ and

$$\mathrm{mdim}\left(I^{\mathbb{N}} \times I^{\mathbb{N}}, f\right) \geq \mathrm{mdim}\left(\{(\ldots, 1, 1, 1, \ldots)\} \times I^{\mathbb{N}}, f\right) = \mathrm{mdim}\left(I^{\mathbb{N}}, \phi\right) > 0.$$

On the other hand, note that $f_k^n(\bar{x}, \bar{y}) \to (\bar{0}, \bar{0})$ as $n \to \infty$ for any $(\bar{x}, \bar{y}) \in I^{\mathbb{N}} \times I^{\mathbb{N}}$ and $k \geq 1$. Hence $\mathrm{mdim}\left(I^{\mathbb{N}} \times I^{\mathbb{N}}, \sigma^k(\boldsymbol{f})\right) = 0$ for any $k \geq 1$, where $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ and therefore $\mathrm{mdim}\left(I^{\mathbb{N}} \times I^{\mathbb{N}}, \boldsymbol{f}\right)^* = 0$.

3 Metric Mean Dimension for Non-autonomous Dynamical Systems

Throughout this section, we will fix $\boldsymbol{f} = (f_n)_{n=1}^{\infty} \in \mathcal{C}(X)$ where X is a compact metric space with metric d. For any $n \in \mathbb{N}$ let $d_n : X \times X \to [0, \infty)$ defined by

$$d_n(x, y) = \max\left\{d(x, y), d(f_1(x), f_1(y)), \ldots, d\left(f_1^{(n-1)}(x), f_1^{(n-1)}(y)\right)\right\}.$$

Thus d_n is a metric on X for all n and generates the same topology induced by d. Fix $\varepsilon > 0$. We say that $A \subset X$ is an $(n, \boldsymbol{f}, \varepsilon)$-*separated* set if $d_n(x, y) > \varepsilon$, for any two distinct points $x, y \in A$. We denote by $\mathrm{sep}(n, \boldsymbol{f}, \varepsilon)$ the maximal cardinality of an $(n, \boldsymbol{f}, \varepsilon)$-separated subset of X. Given an open cover α of X, we say that α is an $(n, \boldsymbol{f}, \varepsilon)$-*cover* if the d_n-diameter of any element of α is less than ε. Let $\mathrm{cov}(n, \boldsymbol{f}, \varepsilon)$ be the minimum number of elements in an $(n, \boldsymbol{f}, \varepsilon)$-cover of X. We say that $E \subset X$ is an $(n, \boldsymbol{f}, \varepsilon)$-*spanning* set for X if for any $x \in X$ there exists $y \in E$ such that $d_n(x, y) < \varepsilon$. Let $\mathrm{span}(n, \boldsymbol{f}, \varepsilon)$ be the minimum cardinality of any $(n, \boldsymbol{f}, \varepsilon)$-spanning subset of X. By the compactness of X, $\mathrm{sep}(n, \boldsymbol{f}, \varepsilon)$, $\mathrm{span}(n, \boldsymbol{f}, \varepsilon)$ and $\mathrm{cov}(n, \boldsymbol{f}, \varepsilon)$ are finite real numbers.

Definition 3.1 We define the *lower metric mean dimension* of $(X, \boldsymbol{f}, d)$ and the *upper metric mean dimension* of $(X, \boldsymbol{f}, d)$ by

$$\underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) = \liminf_{\varepsilon \to 0} \frac{\mathrm{sep}(\boldsymbol{f}, \varepsilon)}{|\log \varepsilon|} \quad \text{and} \quad \overline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) = \limsup_{\varepsilon \to 0} \frac{\mathrm{sep}(\boldsymbol{f}, \varepsilon)}{|\log \varepsilon|},$$

respectively, where $\mathrm{sep}(\boldsymbol{f}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(n, \boldsymbol{f}, \varepsilon)$.

It is not difficult to see that

$$\underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) = \liminf_{\varepsilon \to 0} \frac{\mathrm{span}(\boldsymbol{f}, \varepsilon)}{|\log \varepsilon|} = \liminf_{\varepsilon \to 0} \frac{\mathrm{cov}(X, \varepsilon)}{|\log \varepsilon|},$$

where $\mathrm{span}(\boldsymbol{f}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{span}(n, \boldsymbol{f}, \varepsilon)$ and $\mathrm{cov}(\boldsymbol{f}, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{cov}(n, \boldsymbol{f}, \varepsilon)$. This fact holds for the upper metric mean dimension. We will write $\mathrm{mdim}_M(X, \boldsymbol{f}, d)$ to refer to both $\overline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d)$ and $\underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d)$.

Topological entropy for non-autonomous dynamical systems is invariant under uniform equiconjugacy (see [8] and [15]). Metric mean dimension for single dynamical systems depends on the metric d on X. Consequently, it is not an invariant under conjugacy and therefore it is not an invariant under uniformly equiconjugacy between non-autonomous dynamical systems. Set

$$\mathcal{B} = \{\rho : \rho \text{ is a metric on } X \text{ equivalent to } d\}$$

and take

$$\mathrm{mdim}_M(X, \boldsymbol{f}) = \inf_{\rho \in \mathcal{B}} \mathrm{mdim}_M(X, \boldsymbol{f}, \rho). \tag{3.1}$$

For single maps, $\mathrm{mdim}_M(X, \phi)$ is an invariant under topological conjugacy. In Proposition 6.1 we will prove an analogous result for non-autonomous dynamical systems.

Remark 3.2 It follows from the definition of the topological entropy for non-autonomous dynamical systems introduced in [8] that if the topological entropy of the non-autonomous system $(X, \boldsymbol{f}, d)$ is finite then its metric mean dimension is zero.

Next, we will present some examples of the metric mean dimension for both autonomous and non-autonomous dynamical systems. In Section 5 we will show more examples.

Take $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Consider the metric $\tilde{d}$ on $X^{\mathbb{K}}$ defined by

$$\tilde{d}(\bar{x}, \bar{y}) = \sum_{i\in\mathbb{K}} \frac{1}{2^{|i|}} d(x_i, y_i) \quad \text{for } \bar{x} = (x_i)_{i\in\mathbb{K}}, \bar{y} = (y_i)_{i\in\mathbb{K}} \in X^{\mathbb{K}}. \tag{3.2}$$

Take $X = [0, 1]$, endowed with the metric $d(x, y) = |x - y|$ for $x, y \in X$. In [12], Example E, is proved that $\text{mdim}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right) = 1$. Analogously, we can prove that $\text{mdim}\left(X^{\mathbb{N}}, \sigma, \tilde{d}\right) = 1$:

Lemma 3.3 *Take* $X = [0, 1]$ *endowed with the metric* $d(x, y) = |x - y|$ *for* $x, y \in X$. *Thus*

$$\text{mdim}\left(X^{\mathbb{N}}, \sigma, \tilde{d}\right) = 1.$$

Proof Fix $\varepsilon > 0$ and take $l = \lceil \log(4/\varepsilon) \rceil$, where $\lceil x \rceil = \min\{k \in \mathbb{Z} : x \leq k\}$. Note that $\sum_{n>l} 2^{-n} \leq \varepsilon/2$. Consider the open cover of X given by

$$I_k = \left(\frac{(k-1)\varepsilon}{12}, \frac{(k+1)\varepsilon}{12} \right), \quad \text{for } 0 \leq k \leq \lfloor 12/\varepsilon \rfloor.$$

Note that I_k has length $\varepsilon/6$. Let $n \geq 1$. Next, consider the following open cover of $X^{\mathbb{N}}$:

$$I_{k_1} \times I_{k_2} \times \cdots \times I_{k_{n+l}} \times X \times X \times \cdots, \quad \text{where } 0 \leq k_1, k_2, \ldots, k_{n+l} \leq \lfloor 12/\varepsilon \rfloor.$$

Each open set has diameter less than ε with respect to the distance $\tilde{d}_n$ (see (3.2)). Therefore

$$\text{cov}(n, \sigma, \varepsilon) \leq (1 + \lfloor 12/\varepsilon \rfloor)^{n+l} \leq (2 + 12/\varepsilon)^{n+1+12/\varepsilon}.$$

Hence

$$\text{cov}(\sigma, \varepsilon) = \lim_{n\to\infty} \frac{\log \text{cov}(n, \sigma, \varepsilon)}{n} \leq \lim_{n\to\infty} \frac{(n + 1 + 12/\varepsilon) \log(2 + 12/\varepsilon)}{n} = \log(2+12/\varepsilon).$$

Thus

$$\text{mdim}\left(X^{\mathbb{N}}, \sigma, \tilde{d}\right) = \lim_{\varepsilon\to\infty} \frac{\text{cov}(\sigma, \varepsilon)}{|\log \varepsilon|} \leq 1.$$

On the other hand, any two distinct points in the sets

$$\left\{ (x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} : x_i \in \{0, \varepsilon, 2\varepsilon, \ldots, \lfloor 1/\varepsilon \rfloor \varepsilon\} \text{ for all } 0 \leq i < n \right\}$$

have distance $\geq \varepsilon$ with respect to d_n. It follows that

$$\text{cov}(n, \sigma, \varepsilon) \geq (1 + \lfloor 1/\varepsilon \rfloor)^n \geq (1/\varepsilon)^n.$$

Therefore

$$\text{cov}(\sigma, \varepsilon) \geq \lim_{n\to\infty} \frac{\log \text{cov}(n, \sigma, \varepsilon)}{n} = |\log \varepsilon|.$$

Hence $\text{mdim}_M\left(X^{\mathbb{N}}, \sigma, d\right) = 1.$ □

Next example proves that there exist dynamical systems on the interval with positive metric mean dimension (see also [17]).

Example 3.4 Take $g : [0,1] \to [0,1]$, defined by $x \mapsto |1 - |3x - 1||$, and $0 = a_0 < a_1 < \cdots < a_n < \cdots$, where $a_n = \sum_{k=1}^{n} 6/\pi^2 k^2$ for $n \geq 1$. For each $n \geq 1$, let $T_n : J_n := [a_{n-1}, a_n] \to [0,1]$ be the unique increasing affine map from J_n (which has length $6/\pi^2 n^2$) onto $[0,1]$ and take any strictly increasing sequence of natural numbers m_n. Consider the continuous map $\phi : [0,1] \to [0,1]$ such that, for each $n \geq 1$, $\phi|_{J_n} = T_n^{-1} \circ g^{m_n} \circ T_n$.

Fix $n \geq 1$. Note that J_n can be divided into 3^{m_n} intervals with the same length $J_n(1), \ldots, J_n(3^{m_n})$, such that

$$\phi(J_n(i)) = J_n \quad \text{for each } i \in \left\{1, \ldots, 3^{m_n}\right\}.$$

Next, $J_n(i)$ can be divided into 3^n intervals with the same length $J_n(i,1), \ldots, J_n(i, 3^n)$ such that

$$\phi^2(J_n(i,s)) = J_n \quad \text{for } i = 1, \ldots, 3^n \quad \text{and } s = 1, \ldots, 3^n.$$

Inductively, we can prove that for all $k \geq 1$ and $(i_1, \ldots, i_k)$, where $i_j \in \{1, \ldots, 3^n\}$, we can divide $J_n(i_1, \ldots, i_k)$ into 3^n intervals with the same length $J_n(i_1, \ldots, i_k, 1), \ldots, J_n(i_1, \ldots, i_k, 3^n)$ such that

$$\phi^{k+1}(J_n(i_1, \ldots, i_k, i)) = J_n \quad \text{for } i = 1, \ldots, 3^n.$$

Each $J_n(i_1, \ldots, i_k)$ has length $|J_n|/3^{kn}$ for each $k \geq 1$. Furthermore, each $J_n(i_1, \ldots, i_k)$ has length $|J_n|/3^{km_n}$ for each $k \geq 1$.

Take $\varepsilon_n = |J_n|/3^{m_n} = 3/\pi^2 n^2 3^{m_n}$ for each $n \geq 1$. If $x \in J_n(i_1, \ldots, i_k)$ and $y \in J_n(j_1, \ldots, j_k)$ where $(i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$ and each $i_1, \ldots, i_k, j_1, \ldots, j_k$ is odd, then

$$d_{n+k}(x, y) \geq \varepsilon_n.$$

For each $k \geq 1$, there are more than $(3^{m_n}/2)^k$ intervals $J_n(i_1, \ldots, i_k)$ with i_s odd, $s = 1, \ldots, k$. Hence $\text{sep}(n+k, \phi, \varepsilon_n) \geq (3^{m_n}/2)^k$ and then

$$\text{sep}(\phi, \varepsilon_n) \geq \lim_{k \to \infty} \frac{\log \text{sep}(n+k, \phi, \varepsilon_n)}{k} \geq \log(3^{m_n}/2).$$

Therefore

$$\begin{aligned}\overline{\text{mdim}}_M([0,1], \phi, |\cdot|) &\geq \lim_{n \to \infty} \frac{\log(3^{m_n}/2)}{-\log \varepsilon_n} = \lim_{n \to \infty} \frac{\log(3^{m_n}/2)}{-\log(3/\pi^2 n^2 3^{m_n})} \\ &= \lim_{n \to \infty} \frac{\log(3^{m_n}) + \log 2}{\log(\pi^2 n^2/3) + \log(3^{m_n})} = 1,\end{aligned}$$

hence $\overline{\text{mdim}}_M([0,1], \phi, |\cdot|) \geq 1$. We will obtain from Proposition 5.4 that $\overline{\text{mdim}}_M([0,1], \phi, |\cdot|) \leq 1$. Therefore $\overline{\text{mdim}}_M([0,1], \phi, |\cdot|) = 1$.

Since $\phi(0) = 0$ and $\phi(1) = 1$, the map ϕ induces a continuous map on $\mathbb{S}^1$ with metric mean dimension equal to 1. More generally, we have:

Proposition 3.5 *Take $X = [0,1]$ or $\mathbb{S}^1$. For each $a \in [0,1]$, there exists $\phi_a \in C^0(X)$ with mdim$_M(\phi_a) = a$.*

Proof Any constant map has metric mean dimension equal to 0. On the other hand, Example 3.4 proves that there exist continuous maps on X with metric mean dimension equal to 1. Fix $a \in (0,1)$ and take $r = \frac{1}{a}$. Set $a_0 = 0$ and $a_n = \sum_{i=1}^{n} C\left(3^{-ir}\right)$ for $n \geq 1$, where $C = 1/\sum_{i=1}^{\infty} 3^{-ri} = 1/(3^r - 1)$. For each $n \geq 1$, take J_n, T_n and g as in Example 3.4. Consider the continuous map $\phi_a : [0,1] \to [0,1]$ such that, for each $n \geq 1$, $\phi_a|_{J_n} = T_n^{-1} \circ g^n \circ T_n$ (note that $\phi_a(0) = 0$ and $\phi_a(1) = 1$, consequently ϕ_a induces a continuous map on $\mathbb{S}^1$). Fix

$n \geq 1$. Each J_n can be divided into 3^n intervals with the same length $J_n(1), \ldots, J_n(3^n)$, such that

$$\phi_a(J_n(i)) = J_n \quad \text{for each } i \in \{1, \ldots, 3^n\}.$$

Next, $J_n(i)$ can be divided into 3^n intervals with the same length $J_n(i, 1), \ldots, J_n(i, 3^n)$ such that

$$\phi_a^2(J_n(i, s)) = J_n \quad \text{for } i = 1, \ldots, 3^n \quad \text{and } s = 1, \ldots, 3^n.$$

Inductively, we can prove that for all $k \geq 1$ and $(i_1, \ldots, i_k)$, where $i_j \in \{1, \ldots, 3^n\}$, we can divide $J_n(i_1, \ldots, i_k)$ into 3^n intervals with the same length $J_n(i_1, \ldots, i_k, 1), \ldots, J_n(i_1, \ldots, i_k, 3^n)$ such that

$$\phi_a^{k+1}(J_n(i_1, \ldots, i_k, i)) = J_n \quad \text{for } i = 1, \ldots, 3^n.$$

Each $J_n(i_1, \ldots, i_k)$ has length $|J_n|/3^{kn}$ for each $k \geq 1$.

Take $\varepsilon_n = |J_n| = C/3^{rn}$ for each $n \geq 1$. Each $J_n(i_1, \ldots, i_k)$ has d_{k+1}-diameter equal to ε_n. Consequently, $\text{cov}(k+1, \phi_a, \varepsilon_n) \geq 3^{nk}$ and then

$$\text{cov}(\phi_a, \varepsilon_n) \geq \lim_{k\to\infty} \frac{\log \text{cov}(k+1, \phi_a, \varepsilon_n)}{k+1} \geq \log 3^n.$$

Therefore

$$\begin{aligned} \text{mdim}_M([0,1], \phi_a, |\cdot|) &\geq \lim_{n\to\infty} \frac{\log 3^n}{-\log \varepsilon_n} = \lim_{n\to\infty} \frac{\log 3^n}{-\log(C/3^{nr})} = \lim_{n\to\infty} \frac{\log 3^n}{\log 3^{nr}} \\ &= \lim_{n\to\infty} \frac{n \log 3}{nr \log 3} = \frac{1}{r} = a. \end{aligned}$$

On the other hand, fix $n \geq 1$. Let $m \geq n$ be such that $\sum_{i=m}^{\infty} C\left(3^{-ir}\right) < \varepsilon_n$. Therefore

$$\text{cov}\left(\cup_{i=m}^{\infty} J_i, k, \phi_a, \varepsilon_n\right) = 1 \quad \text{for any } k \geq 1. \tag{3.3}$$

Note that for each $k \geq 1$ and $(i_1, \ldots, i_k)$, where $i_j \in \{1, \ldots, 3^n\}$, the subintervals $J_n(i_1, \ldots, i_k)$ have diameter less than ε_n with the metric d_k for any $k \geq 1$. Consequently, we have

$$\text{cov}(J_n, k, \phi_a, \varepsilon_n) \leq \left(3^n\right)^k \quad \text{for any } k \geq 1. \tag{3.4}$$

For each $i \in \{1, \ldots, n-1\}$, divide each interval J_i into $(3^n)^{k+1}\lceil |J_i|/|J_n| \rceil$ subintervals with the same length, where $\lceil x \rceil = \min\{j \in \mathbb{Z} : x \leq j\}$. Each subinterval has d_k-diameter less than ε_n, thus

$$\text{cov}\left(\cup_{i=1}^{n-1} J_i, k, \phi_a, \varepsilon_n\right) \leq \sum_{i=1}^{n-1} \left(3^n\right)^{k+1} \lceil |J_i|/|J_n| \rceil. \tag{3.5}$$

For $i \in \{n+1, \ldots, m-1\}$, each J_i has d_k-diameter less than ε_n, thus

$$\text{cov}\left(\cup_{i=n+1}^{m-1} J_i, k, \phi_a, \varepsilon_n\right) \leq m - n - 1 \quad \text{for any } k \geq 1. \tag{3.6}$$

By (3.3)–(3.6), we have

$$\begin{aligned} \text{cov}(\phi_a, \varepsilon_n) &\leq \lim_{k\to\infty} \frac{\log\left[1 + (3^n)^k + \sum_{i=1}^{n-1} (3^n)^{k+1} \lceil |J_i|/|J_n| \rceil + m - n - 1\right]}{k-1} \\ &\leq \lim_{k\to\infty} \frac{\log\left[\left(\sum_{i=1}^{n-1} \lceil |J_i|/|J_n| \rceil + m - n + 1\right) (3^n)^{k+1}\right]}{k-1} = \lim_{k\to\infty} \frac{\log(3^n)^{k+1}}{k-1} \\ &= \log\left(3^n\right). \end{aligned}$$

Hence

$$\mathrm{mdim}_M([0,1],\phi_a,|\cdot|) \leq \lim_{n\to\infty}\frac{\log(3^n)}{-\log\varepsilon_n} = \lim_{n\to\infty}\frac{\log(3^n)}{\log(3^{rn})} = \lim_{n\to\infty}\frac{n\log(3)}{(rn)\log(3)} = a.$$

Therefore $\mathrm{mdim}_M([0,1],\phi_a,|\cdot|) = a$. □

Example 3.6 Let $X = \{0,1\}^{\mathbb{N}}$ with its usual metric and consider $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$, where $f_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is given by $f_n(\omega) = \sigma^{2^n}(\omega)$, for any $n \in \mathbb{N}$. Note that $f_1^{(n)}(\omega) = \sigma^{2^{n+1}-2}(\omega)$. We claim that $\mathrm{mdim}_M(X,\boldsymbol{f},d) = \infty$. Fix $\varepsilon > 0$. Take a positive integer k so that $2^{-(k+1)} \leq \varepsilon < 2^{-k}$. Now consider $A \subset \{0,1\}^{\mathbb{N}}$ a $(2^{n+1}-2,\varepsilon)$-separated set for the shift map σ of maximum cardinality and note that A is an (n,ε)-separated set for $\boldsymbol{f}$. Therefore, $\mathrm{sep}(n,\boldsymbol{f},\varepsilon) \geq 2^{2^{n+1}-2+k}$ and then

$$\frac{\log \mathrm{sep}(n,\boldsymbol{f},\varepsilon)}{n\log\varepsilon} \geq \frac{(2^{n+1}-2+k)\log 2}{nk}.$$

Hence, by the definition of the upper metric mean dimension, we have

$$\mathrm{mdim}_M(X,\boldsymbol{f},d) = \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log \mathrm{sep}(n,\boldsymbol{f},\varepsilon)}{n|\log\varepsilon|} = \infty.$$

In [19], Zhu, Liu, Xu, and Zhang showed that if X is a k-dimensional Riemannian manifold and $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ is a sequence of C^1-maps on X such that $a_n = \sup_{x\in M}\|D_x f_n\| < \infty$ for all $n \in \mathbb{N}$, then

$$h_{top}(\boldsymbol{f}) \leq \max\left\{0, \limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log a_i\right\}.$$

Hence, by Remark 3.2, we have:

Proposition 3.7 *If* $\limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log a_i < \infty$, *we have* $\mathrm{mdim}_M(M,\boldsymbol{f},d) = 0$.

Any sequence of homeomorphisms on both the interval or the circle has zero topological entropy (see [8], Theorem D). Therefore, the metric mean dimension of any $\boldsymbol{f}$ on both the interval or the circle is equal to zero. In the next example we will see that there exist non-autonomous dynamical systems consisting of diffeomorphisms on a surface with infinite metric mean dimension.

Example 3.8 Let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism induced by a hyperbolic matrix A with eigenvalue $\lambda > 1$, where $\mathbb{T}^2$ is the torus endowed with the metric d inherited from the plane. Consider $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ where $f_n = \phi^{2^n}$ for each $i \geq 1$. We have $|\mathrm{Fix}(\phi^n)| = \lambda^n + \lambda^{-n} - 2$, where $\mathrm{Fix}(\psi)$ is the set consisting of fixed points of a continuous map ψ (see [5], Proposition 1.8.1). Furthermore,

$$\mathrm{sep}(n,\boldsymbol{f},1/4) \geq \mathrm{sep}\left(2^n,\phi,1/4\right) \geq \mathrm{Fix}\left(\phi^{2^n}\right) = \lambda^{2^n} + \lambda^{-2^n} - 2$$

(see [5], Chapter 3, Section 2.e). Therefore,

$$\lim_{n\to\infty}\frac{\mathrm{sep}(n,\boldsymbol{f},1/4)}{n} \geq \lim_{n\to\infty}\frac{\log\lambda^{2^n}}{n} = \infty,$$

and hence $\mathrm{mdim}_M\left(\mathbb{T}^2,\boldsymbol{f},d\right) = \infty$.

Suppose the Hausdorff dimension of X is finite. Let $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ be a non-autonomous dynamical system where each f_n is a C^r-map on X. We have that if $h_{top}(\boldsymbol{f}) < \infty$ then $\text{mdim}_M(X,\boldsymbol{f}, d) = 0$. Therefore, if $\sup_{n\in\mathbb{N}} L(f_n) < \infty$, where $L(f_n)$ is the Lipschitz constant of f_n, we have that $h_{top}(\boldsymbol{f}) < \infty$ and hence $\text{mdim}_M(X,\boldsymbol{f}, d) = 0$. Thus if $\sup_{n\in\mathbb{N}} L(f_n) < \infty$, then $\text{mdim}_M(X,\boldsymbol{f}, d) = 0$. In particular, if X is a compact Riemannian manifold and $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ is a sequence of differentiable maps that $\sup_{n\in\mathbb{N}} \|Df_n\| < \infty$, where Df_n is the derivative of f_n, we have that $h_{top}(\boldsymbol{f}) < \infty$ and hence $\text{mdim}_M(X,\boldsymbol{f}, d) = 0$.

4 Some Fundamental Properties of the Metric Mean Dimension

In this section we show some properties which are well-known for topological entropy and metric mean dimension for dynamical systems. In the next proposition we will consider $\boldsymbol{f}^{(p)}$, which was defined in Definition 2.3.

It is well-known that $h_{top}(\boldsymbol{f}^{(p)}) \leq p\, h_{top}(\boldsymbol{f})$ and if the sequence $(f_n)_{n=1}^{\infty}$ is equicontinuous, then the equality holds (see [8], Lemma 4.2). For the case of the metric mean dimension, we always have that $\text{mdim}_M\left(X,\boldsymbol{f}^{(p)}, d\right) \leq p\,\text{mdim}_M(X,\boldsymbol{f}, d)$. However we will present an example where the inequality can be strict even for single continuous maps (see Remark 4.2).

Proposition 4.1 *For any $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ and $p \in \mathbb{N}$, we have*

$$\text{mdim}_M\left(X,\boldsymbol{f}^{(p)}, d\right) \leq p\,\text{mdim}_M(X,\boldsymbol{f}, d).$$

Consequently (see (3.1)),

$$\text{mdim}_M\left(X,\boldsymbol{f}^{(p)}\right) \leq p\,\text{mdim}_M(X,\boldsymbol{f}).$$

Proof Note that, for any positive integer m, we have

$$\max_{0\leq j<m} d\left(f_1^{(jp)}(x), f_1^{(jp)}(y)\right) \leq \max_{0\leq j<mp} d\left(f_1^{(j)}(x), f_1^{(j)}(y)\right).$$

Thus $\text{span}\left(m,\boldsymbol{f}^{(p)}, \varepsilon\right) \leq \text{span}(mp,\boldsymbol{f}, \varepsilon)$ and therefore

$$\text{span}\left(\boldsymbol{f}^{(p)}, \varepsilon\right) = \limsup_{m\to\infty} \frac{1}{m} \log \text{span}\left(m,\boldsymbol{f}^{(p)}, \varepsilon\right) \leq p \limsup_{m\to\infty} \frac{1}{mp} \log \text{span}(m,\boldsymbol{f}, \varepsilon) = p\, \text{span}(\boldsymbol{f}, \varepsilon).$$

Hence $\text{mdim}_M\left(X,\boldsymbol{f}^{(p)}, d\right) \leq p\,\text{mdim}_M(X,\boldsymbol{f}, d)$. □

Remark 4.2 In Example 3.4 we prove that there exists a continuous map $\phi : [0, 1] \to [0, 1]$ such that $\underline{\text{mdim}}_M([0, 1], \phi, d) = 1$, where $d(x, y) = |x - y|$ for $x, y \in [0, 1]$. It follows from Proposition 5.4 that for any $f : [0, 1] \to [0, 1]$ we have $\overline{\text{mdim}}_M([0, 1], f, d) \leq 1$. Consequently, $\overline{\text{mdim}}_M([0, 1], \phi^n, d) \leq 1$ for any $n \geq 1$, which proves that the inequality in Proposition 4.1 can be strict for autonomous systems and therefore for non-autonomous systems.

If $A, B \subseteq X$ are invariant subsets under a continuous map ϕ, then

$$h_{top}(\phi) = \max\{h_{top}(\phi|_A), h_{top}(\phi|_B)\}.$$

It is clear this property is also valid for the metric mean dimension.

Proposition 4.3 *If $A, B \subseteq X$ are invariant subsets under ϕ, then*

$$\mathrm{mdim}_M(X, \phi, d) = \max\{\mathrm{mdim}_M(X, \phi|_A, d), \mathrm{mdim}_M(X, \phi|_B, d)\}.$$

If $A_1, A_2, \ldots$ is a sequence of invariant subsets under ϕ, then

$$\max_{n\in\mathbb{N}} \{\mathrm{mdim}_M(X, \phi|_{A_n}, d)\} \leq \mathrm{mdim}_M(X, \phi, d).$$

Example 3.4 proves that the inequality can be strict (the sets $J_1, J_2, \ldots$ are invariant under ϕ, however $\mathrm{mdim}_M(X, \phi|_{J_n}, d) = 0$ for each n).

Metric mean dimension can be defined on any subset A of X. Kolyada and Snoha in [8], Lemma 4.1, proved that if $X = \cup_i^n A_i$, then

$$h_{top}(\boldsymbol{f}) = \max_{i=1,\ldots,n} h_{top}\left(\boldsymbol{f}|_{A_i}\right).$$

Analogously we can prove that:

Proposition 4.4 *If $X = \cup_i^n A_i$, then*

$$\mathrm{mdim}_M(X,\boldsymbol{f}, d) = \max_{i=1,\ldots,n} \mathrm{mdim}_M(X,\boldsymbol{f}|_{A_i}, d).$$

Definition 4.5 We say that $x \in X$ is a *nonwandering point* for $\boldsymbol{f}$ if for every neighborhood U of x there exist positive integers k and n with $f_n^{(k)}(U) \cap U \neq \emptyset$. We denote by $\Omega(\boldsymbol{f})$ the set consisting of the nonwandering points of $\boldsymbol{f}$.

It is well-known that for any continuous map $\phi : X \to X$ we have $h_{top}(\phi) = h_{top}(\phi|_{\Omega(\phi)})$. This fact was proved for non-autonomous dynamical systems by Kolyada and Snoha in [8]. For mean dimension of single continuous maps this fact was proved by Gutman in [3], Lemma 7.2. For the metric mean dimension of non-autonomous dynamical systems we also have:

Theorem 4.6 *We have*

$$\mathrm{mdim}_M(X,\boldsymbol{f}, d) = \mathrm{mdim}_M(\Omega(\boldsymbol{f}),\boldsymbol{f}, d).$$

Proof It is clear that $\mathrm{mdim}_M(X,\boldsymbol{f}, d) \geq \mathrm{mdim}_M(\Omega(\boldsymbol{f}),\boldsymbol{f}, d)$. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let α be an open $(n,\boldsymbol{f}, \varepsilon)$-cover of X with minimum cardinality. Take β a minimal finite open subcover of $\Omega(\boldsymbol{f})$, chosen from α (note that β is an $(n,\boldsymbol{f}, \varepsilon)$-cover of $\Omega(\boldsymbol{f})$). By the minimality of α we have that β is an $(n,\boldsymbol{f}, \varepsilon)$-cover of $\Omega(\boldsymbol{f})$ with minimum cardinality, which we denote by $\mathrm{cov}(\Omega(\boldsymbol{f}), n,\boldsymbol{f}, \varepsilon)$, i.e., $\mathrm{Card}(\beta) = \mathrm{cov}(\Omega(\boldsymbol{f}), n,\boldsymbol{f}, \varepsilon)$.

The set $K = X\setminus \bigcup_{U\in\beta} U$ is compact and consists of wandering points. We can cover K by a finite number of wandering subsets, each of them contained in some element of α. The sets defined before together with β form a finite open cover $\gamma(n) = \gamma$ of X, finer than α. Consider, for each k, the open cover $\gamma\left(k,\boldsymbol{f}^{(n)}\right)$ associated with the sequence $\boldsymbol{f}^{(n)}$. Note that each element of $\gamma\left(k,\boldsymbol{f}^{(n)}\right)$ is of the form

$$A_0\cap\left(f_1^{(n)}\right)^{-1}(A_1)\cap\left(f_1^{(n)}\right)\circ\left(f_{n+1}^{(n)}\right)^{-1}(A_2)\cap\cdots\cap\left(f_1^{(n)}\right)^{-1}\circ\cdots\circ\left(f_{(k-2)n+1}^{(n)}\right)^{-1}(A_{k-1}),$$

where $A_i \in \gamma$, for $i = 0 \dots, k-1$. It implies that $\gamma\left(k, \boldsymbol{f}^{(n)}\right)$ is a $\left(k, \boldsymbol{f}^{(n)}, \varepsilon\right)$-cover of X. Let A_i and A_j be non-empty open sets of $\gamma\left(k, \boldsymbol{f}^{(n)}\right)$ for some $i < j$. If $A_i = A_j$, then

$$\left(f_{(j-1)n+1}^{(n)} \circ \cdots \circ f_{in+1}^{(n)}\right)(A_i) = f_{in+1}^{(j-i)n}(A_i)$$

intersects $A_i = A_j$. In that case A_i does not contain non-wandering points for $\boldsymbol{f}$ (and hence $A_i \in \beta$). Now we estimate the number of elements of $\gamma\left(k, \boldsymbol{f}^{(n)}\right)$. Setting

$$j := \text{Card}\{A_i : i = 0, 1, \dots k-1\} \quad \text{and} \quad m := \text{Card}(\gamma\left(k, \boldsymbol{f}^{(n)}\right) \backslash \beta),$$

we have $0 \le j \le m$. In this case we have $\binom{m}{j}$ possibilities of the choice of a j-element subset of $\gamma\left(k, \boldsymbol{f}^{(n)}\right) \backslash \beta$ and then these sets can appear as various A_i's in $k \cdot (k-1) \cdots (k - j + 1) = k!/(k-j)!$ ways. For the rest of A_i's we can choice any element of β. So, the number of elements of $\gamma\left(k, \boldsymbol{f}^{(n)}\right)$ is bounded by

$$\sum_{j=0}^{m} \binom{m}{j} \frac{k!}{(k-j)!} \cdot (\text{Card}(\beta))^{k-j}.$$

Since $k!/(k-j)! \le k^m$ and $\binom{m}{j} \le m!$, this number is not larger than $(m+1) \cdot m! \cdot k^m \cdot (\text{Card}(\beta))^k$. Thus, using the fact that $\text{cov}\left(k, \boldsymbol{f}^{(n)}, \varepsilon\right) \le \text{Card}\left(\gamma\left(k, \boldsymbol{f}^{(n)}\right)\right)$, we have

$$\limsup_{k\to\infty} \frac{1}{k} \log \text{cov}\left(k, \boldsymbol{f}^{(n)}, \varepsilon\right) \le \limsup_{k\to\infty} \frac{1}{k} \log(m+1) \cdot m! \cdot k^m \cdot (\text{Card}(\beta))^k = \log(\text{Card}(\beta)).$$

As

$$\limsup_{k\to\infty} \frac{1}{k} \log \text{cov}\left(k, \boldsymbol{f}^{(n)}, \varepsilon\right) = n \limsup_{k\to\infty} \frac{1}{k} \log \text{cov}(k, \boldsymbol{f}, \varepsilon),$$

it follows that

$$\limsup_{k\to\infty} \frac{1}{k} \log \text{cov}(k, \boldsymbol{f}, \varepsilon) \le \tfrac{1}{n} \log \text{cov}(\Omega(\boldsymbol{f}), n, \boldsymbol{f}, \varepsilon).$$

Taking the limsup as $n \to \infty$ we obtain

$$\text{cov}(\boldsymbol{f}, \varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log \text{cov}(\Omega(\boldsymbol{f}), n, \boldsymbol{f}, \varepsilon) := \text{cov}(\Omega(\boldsymbol{f}), \boldsymbol{f}, \varepsilon).$$

So,

$$\text{mdim}_M(X, \boldsymbol{f}, d) = \liminf_{\varepsilon\to 0} \tfrac{\text{cov}(\boldsymbol{f}, \varepsilon)}{|\log \varepsilon|} \le \liminf_{\varepsilon\to 0} \tfrac{\text{cov}(\Omega(\boldsymbol{f}), \boldsymbol{f}, \varepsilon)}{|\log \varepsilon|} = \text{mdim}_M(\Omega(\boldsymbol{f}), \boldsymbol{f}, d),$$

which proves the theorem. □

Definition 4.7 A continuous map $\psi : X \to Y$ will be called *α-compatible* if it is possible to find a finite open cover β of $\psi(X)$ such that $\psi^{-1}(\beta) \succ \alpha$.

Lindenstrauss and Weiss in [11], Theorem 4.2, proved that for any metric d compatible with the topology of X, we have

$$\mathrm{mdim}(X, \phi) \leq \underline{\mathrm{mdim}_M}(X, \phi, d)$$

for any continuous map $\phi : X \to X$. These ideas work in order to show the non-autonomous case: metric mean dimension is an upper bound for the mean dimension of non-autonomous dynamical systems. We will need the next proposition, whose proof can be found in [11], Proposition 2.4.

Proposition 4.8 *If α is an open cover of X, then $\mathcal{D}(\alpha) \leq k$ if and only if there exists an α-compatible continuous map $\psi : X \to K$, where K has topological dimension k.*

Theorem 4.9 *For any metric d on X compatible with the topology of X we have that*

$$\mathrm{mdim}(X, \boldsymbol{f}) \leq \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d).$$

Proof Let α be an open cover of X. We can assume that α is of the form

$$\alpha = \{U_1, V_1\} \vee \cdots \vee \{U_\ell, V_\ell\},$$

where each $\{U_i, V_i\}$ is an open cover of X with two elements. For each $1 \leq i \leq \ell$ define $\omega_i : X \to [0, 1]$ by

$$\omega_i(x) = \frac{d(x, X \backslash V_i)}{d(x, X \backslash U_i) + d(x, X \backslash V_i)}.$$

It is not difficult to see that ω_i is Lipschitz, $U_i = \omega_i^{-1}([0, 1))$ and $V_i = \omega_i^{-1}((0, 1])$.

Let C be a common bound for the Lipschitz constants of all ω_i. For each positive integer N define $F(N, \cdot) : X \to [0, 1]^{\ell N}$ by

$$F(N, x) = (\omega_1(x), \ldots, \omega_\ell(x), \omega_1(f_1(x)), \ldots, \omega_\ell(f_1(x)), \ldots, \omega_1\left(f_1^{(N)}(x)\right), \ldots, \omega_\ell\left(f_1^{(N)}(x))\right).$$

As $U_i = \omega_i^{-1}([0, 1))$ and $V_i = \omega_i^{-1}((0, 1])$ we have that $F(N, \cdot) \succ \alpha_0^{N-1}$.

Now for each $S \subset \{1, \ldots, \ell N\}$, for $x \in X$, denote by $F(N, x)_S$ the projection of $F(N, x)$ to the coordinates of the index set S.

Claim Let $\varepsilon > 0$ and $D = \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d)$. There exists $N(\varepsilon) > 0$ so that, for all $N > N(\varepsilon)$ there exists $\xi \in (0, 1)^{\ell N}$ which satisfies

$$\xi_S \notin F(N, X)_S,$$

for any subset $S \subset \{1, \ldots, \ell N\}$ that satisfies $|S| > (D + \varepsilon)N$.

Proof Let $\delta > 0$ such that

$$\delta < \left(2^\ell (2C)^{2D}\right)^{-2\varepsilon} \qquad \text{and} \qquad \frac{\mathrm{sep}(\boldsymbol{f}, \delta)}{\log \delta} = \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) + \frac{\varepsilon}{4}.$$

We notice that for N sufficiently large we can cover X by $\delta^{-(D+\varepsilon/2)N}$ dynamical balls $B(x, N, \delta) = \{y \in X : d_N(x, y) < \delta\}$. Since C is the common Lipschitz constant for all ω_i, we conclude that

$$F(N, B(x, N, \delta)) \subset \{a \in [0, 1]^{\ell N} : \|F(N, x) - a\|_\infty < C\delta\},$$

where $\|(a_1, \ldots, a_{\ell N}) - (b_1, \ldots, b_{\ell N})\|_\infty = \sup_i |a_i - b_i|$. This fact implies that $F(N, X)$ can be covered by $\delta^{-(D+\varepsilon/2)N}$ balls in the $\|\ \cdot\|_\infty$ norm of radius $C\delta$. Let $B(1), \ldots, B(K)$ be these balls, with $K = \delta^{-(D+\varepsilon/2)N}$.

Choose $\xi \in [0, 1]^{\ell N}$ with uniform probability and notice that

$$\mathbb{P}(\xi \in F(N, X)_S) \leq \sum_{j=1}^{K} \mathbb{P}(\xi \in B(j)_S) \leq \delta^{-(D+\varepsilon/2)N}(2C\delta)^{|S|},$$

and so

$$\begin{aligned} \mathbb{P}(\exists S : |S| > (D+\varepsilon)N \text{ and } \xi_S \in F(N, X)_S) &\leq \sum_{|S|>(D+\varepsilon)N} \mathbb{P}(\xi_S \in F(N, X)_S) \\ &\leq (\sharp \text{ of such } S)\delta^{-(D+\varepsilon/2)N}(2C\delta)^{D+\varepsilon}N \\ &\leq 2^{\ell N}\left((2C)^{2D}\delta^{\varepsilon/2}\right)^N \ll 1. \end{aligned}$$

Hence, with high probability, a random ξ will satisfy the requirements. □

Claim If $\pi : F(N, X) \to [0, 1]^{\ell N}$ satisfies for both $a = 0$ and $a = 1$, and all $\xi \in [0, 1]^{\ell N}$,

$$\{1 \leq k \leq \ell N : \xi_k = a\} \subset \{1 \leq k \leq \ell N : \pi(\xi)_S = a\},$$

then $\pi \circ F(N, X)$ is compatible with α_0^{N-1}.

Proof Given $\xi \in [0, 1]^{\ell N}$, define for $0 \leq j < N$ and $1 \leq i < \ell$

$$W_{i,j} = \begin{cases} \left(f_1^{(j)}\right)^{-1}(U_i), & \text{if } \xi_{j\ell+i} = 0, \\ \left(f_1^{(j)}\right)^{-1}(V_i), & \text{otherwise.} \end{cases}$$

By the definition of $W_{i,j}$ we have that $(\pi \circ F(N, \cdot))^{-1}(\xi) \subset \bigcap_{1\leq i\leq \ell, 0\leq j<N} W_{i,j} \in \alpha_0^{N-1}$. It follows that $\pi \circ F(N, X)$ is compatible with α_0^{N-1}. □

For a fixed $\varepsilon > 0$, consider $\bar{\xi}$ and N as in the first Claim. Set

$$\Phi = \left\{\xi \in [0, 1]^{\ell N} : \xi_k = \bar{\xi}_k \text{ for more than } (D+\varepsilon)N \text{ indexes } k\right\}.$$

Then, $F(N, X) \subset \Phi^C = [0, 1]^{\ell N} \backslash \Phi$.

Now, for each $m = 1, 2, \ldots,$ denote by J_m the set

$$J_m = \left\{\xi \in [0, 1]^{\ell N} : \xi_i \in \{0, 1\} \text{ for at least } m \text{ indexes } 1 \leq i \leq \ell N\right\}.$$

Since $\bar{\xi}$ is in the interior of $[0, 1]^{\ell N}$, one can define $\pi_1 : [0, 1]^{\ell N} \backslash \{\bar{\xi}\} \to J_1$ by mapping each ξ to the intersection of the ray starting at $\bar{\xi}$ and passing through ξ and J_1. For each of the $(\ell N - 1)$-dimensional cubes I^t that comprises J_1 we can define a retraction on I^t in a similar fashion using as a center the projection of $\bar{\xi}$ on I^t. This will define a continuous retraction π_2 of Φ^C onto J_2. As long as there is some intersection of Φ with the cubes in J_m this process can be continued, thus we finally get a continuous projection π of Φ^C onto J_{m_0}, a space of topological dimension equals to m_0, with

$$m_0 \leq \lfloor D + \varepsilon \rfloor N + 1,$$

where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$. By construction, π satisfies the hypotheses of the second claim. Thus $\pi \circ F(N, \cdot) \succ \alpha_0^{N-1}$. Moreover, since $F(N, X) \subset \Phi^C$, we have $\pi(F(N, X)) \subset J_{m_0}$.

Putting all together, we have constructed a α_0^{N-1} compatible function from X to a space of topological dimension less or equal to $\lfloor D + \varepsilon \rfloor N + 1$, and so

$$\frac{D\left(\alpha_0^{N-1}\right)}{N} \leq \frac{\lfloor D + \varepsilon \rfloor N + 1}{N}.$$

As ε goes to zero we get that $\mathrm{mdim}(X, \boldsymbol{f}) \leq D$. $\square$

The inequality in the theorem above can be strict for single maps and therefore for non-autonomous dynamical systems. In [10], Theorem 4.3, is proved that if a continuous map $\phi : X \to X$ is an extension of a minimal system, then there is a metric d' on X, equivalent to d, such that

$$\mathrm{mdim}(X, \phi) = \underline{\mathrm{mdim}}_M(X, \phi, d').$$

5 Upper Bound for the Metric Mean Dimension

As we saw in Remark 2.2, we have $\mathrm{mdim}\left(X^{\mathbb{K}}, \sigma\right) \leq \dim(X)$, where $\mathbb{K} = \mathbb{Z}$ or $\mathbb{N}$. Furthermore, if $X = I^k$, then $\mathrm{mdim}\left(X^{\mathbb{Z}}, \sigma\right) = k$. In this section we will prove that the metric mean dimension of the shift on $X^{\mathbb{K}}$ is equal to the box dimension of X with respect to the metric d, which will be defined below. This fact implies that the metric mean dimension of any continuous map $\phi : X \to X$ is less or equal to the box dimension of X with respect to the metric d (see Proposition 5.4).

Definition 5.1 For $\varepsilon > 0$, let $N(\varepsilon)$ be the minimum number of closed balls of radious ε needed to cover X. The numbers

$$\overline{\dim_B}(X, d) = \limsup_{\varepsilon \to \infty} \frac{\log N(\varepsilon)}{|\log \varepsilon|} \quad \text{and} \quad \underline{\dim_B}(X, d) = \liminf_{\varepsilon \to \infty} \frac{\log N(\varepsilon)}{|\log \varepsilon|}$$

are called, respectively, the *upper Minkowski dimension* (or *upper box dimension*) of X and the *lower Minkowski dimension* (or *lower box dimension*) of X, with respect to d.

For any metric space (X, d) we have

$$\dim(X) \leq \dim_H(X, d) \leq \underline{\dim_B}(X, d),$$

where $\dim_H(X, d)$ is the Hausdorff dimension of X with respect to d (see [6], Section II, A). If $X = [0, 1]$, then $\dim(X) = \dim_H(X, d) = \underline{\dim_B}(X, d) = 1$. However, there exist sets such that the inequalities above can be strict, as we will see in the next example, which also proves that neither $\dim(X)$ nor $\dim_H(X, d)$ are upper bounds for $\overline{\mathrm{mdim}_M}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right)$.

Example 5.2 Let $A = \{0\} \cup \{1/n : n \geq 1\}$ endowed with the metric $d(x, y) = |x - y|$ for $x, y \in A$. In [6], Lemma 3.1, is proved that $\dim_H(A) = 0$ while $\underline{\dim_B}(A) = 1/2$. Furthermore, we have

$$\underline{\mathrm{mdim}_M}\left(A^{\mathbb{Z}}, \sigma, \tilde{d}\right) = \underline{\dim_B}(A) = 1/2$$

(see [12], Section VII).

Using the *Classical Variational Principle*, in [17], Theorem 5, the authors claim to have proven that for any (X, d)

$$\overline{\text{mdim}_M}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right) = \overline{\dim_B}(X, d).$$

This fact can be proved generalizing the ideas given in [12], Example E:

Theorem 5.3 *For* $\mathbb{K} = \mathbb{Z}$ *or* $\mathbb{N}$ *we have*

$$\overline{\text{mdim}_M}\left(X^{\mathbb{K}}, \sigma, \tilde{d}\right) = \overline{\dim_B}(X, d) \qquad \textit{and} \qquad \underline{\text{mdim}_M}\left(X^{\mathbb{K}}, \sigma, \tilde{d}\right) = \underline{\dim_B}(X, d).$$

Proof We will prove the case $\mathbb{K} = \mathbb{Z}$ (the case $\mathbb{K} = \mathbb{N}$ can be proved analogously as in Lemma 3.3). Fix $\varepsilon > 0$ and take l big enough such that $\sum_{n>l} 2^{-n}\text{diam}(X) \leq \varepsilon/2$. Let $m = N(\varepsilon)$ be the minimum number of closed ε-balls $X_1, \ldots, X_m$ needed to cover X. Consider the open cover of $X^{\mathbb{Z}}$ given by the open sets

$$\cdots \times X \times X_{k_{-l}} \times X_{k_{-l+1}} \times \cdots \times X_{k_{n+l}} \times X \times \cdots, \qquad \text{where } 1 \leq k_{-l}, k_{-l+1}, \ldots, k_{n+l} \leq m.$$

Note that each one of these open sets has diameter less than 4ε with respect to the distance $\tilde{d}_n$ on $X^{\mathbb{Z}}$. Therefore $\text{cov}(n, \sigma, 4\varepsilon) \leq m^{n+2l+1}$ and hence

$$\text{cov}(\sigma, 4\varepsilon) = \lim_{n\to\infty} \frac{\log \text{cov}(n, \sigma, 4\varepsilon)}{n} \leq \lim_{n\to\infty} \frac{(n + 2l + 1)\log(m)}{n} = \log N(\varepsilon),$$

which implies that

$$\overline{\text{mdim}_M}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right) = \limsup_{\varepsilon\to\infty} \frac{\text{cov}(\sigma, 4\varepsilon)}{|\log 4\varepsilon|} \leq \limsup_{\varepsilon\to\infty} \frac{\log N(\varepsilon)}{|\log 4\varepsilon|} = \limsup_{\varepsilon\to\infty} \frac{\log N(\varepsilon)}{|\log 4 + \log \varepsilon|} = \overline{\dim_B}(X, d)$$

and

$$\underline{\text{mdim}_M}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right) = \liminf_{\varepsilon\to\infty} \frac{\text{cov}(\sigma, 4\varepsilon)}{|\log 4\varepsilon|} \leq \underline{\dim_B}(X, d),$$

To prove the converse inequality, for $\varepsilon > 0$ let $\{x_1, x_2, \ldots, x_{N(\varepsilon)}\}$ be a maximal set of points in X which are ε-separated. For $n \geq 1$, consider the set

$$\left\{(y_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}} : y_i \in \left\{x_1, x_2, \ldots, x_{N(\varepsilon)}\right\} \text{ for all } -l \leq i \leq n + l\right\}$$

and notice that it is (σ, n, ε)-separated and its cardinality is bounded from below by $N(\varepsilon)^{n+2l+1}$. So

$$\text{sep}(\sigma, \varepsilon) \geq \lim_{n\to\infty} \frac{\log N(\varepsilon)^{n+2l+1}}{n} = \log N(\varepsilon),$$

and it implies that

$$\underline{\text{mdim}_M}\left(X^{\mathbb{Z}}, \sigma, \tilde{d}\right) \geq \underline{\dim_B}(X, d),$$

which proves the theorem. □

Next proposition proves the metric mean dimension of any dynamical system is bounded by the box dimension of the space (see [17], Remark 4).

Proposition 5.4 *For any continuous map* $\phi : X \to X$ *we have*

$$\overline{\text{mdim}_M}(X, \phi, d) \leq \overline{\dim_B}(X, d) \qquad \textit{and} \qquad \underline{\text{mdim}_M}(X, \phi, d) \leq \underline{\dim_B}(X, d).$$

In particular, if $X = [0, 1]$*, then*

$$\underline{\text{mdim}_M}(X, \phi, d) \leq \overline{\text{mdim}_M}(X, \phi, d) \leq 1.$$

Proof Consider the embedding $\psi : X \to X^{\mathbb{N}}$, defined by $x \mapsto \psi(x) = \big(x, \phi(x), \phi^2(x), \dots\big)$. We have $\sigma \circ \psi = \psi \circ \phi$. Therefore, $Y = \psi(X)$ is a closed subset of $X^{\mathbb{N}}$ invariant by σ. Take the metric d_ψ on X defined by $d_\psi(x, y) = \tilde{d}(\psi(x), \psi(y))$, for any $x, y \in X$. Clearly $d(x, y) \leq d_\psi(x, y)$ for any $x, y \in X$, therefore any (n, ϕ, ε)-separated subset of X with respect to d is a (n, ϕ, ε)-separated subset of X with respect to d_ψ. Hence

$$\overline{\mathrm{mdim}_M}(X, \phi, d) \leq \overline{\mathrm{mdim}_M}(X, \phi, d_\psi) = \overline{\mathrm{mdim}_M}\left(Y, \sigma|_Y, \tilde{d}\right) \leq \overline{\mathrm{mdim}_M}\left(X^{\mathbb{N}}, \sigma, \tilde{d}\right) \leq \overline{\dim_B}(X, d)$$

and, analogously, $\underline{\mathrm{mdim}_M}(X, \phi, d) \leq \underline{\dim_B}(X, d)$. □

Example 3.4 proves that there exist dynamical systems $\phi : X \to X$ such that

$$\overline{\mathrm{mdim}_M}(X, \phi, d) = \overline{\dim_B}(X, d) \quad \text{and} \quad \underline{\mathrm{mdim}_M}(X, \phi, d) = \underline{\dim_B}(X, d).$$

We can consider the *asymptotic metric mean dimension* as the limit

$$\mathrm{mdim}_M(X, \boldsymbol{f}, d)^* = \limsup_{i \to \infty} \mathrm{mdim}_M\left(X, \sigma^i(\boldsymbol{f}), d\right).$$

Theorem 5.5 *If $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ converges uniformly to a continuous map $f : X \to X$, then, for any $k \geq 1$,*

$$\mathrm{mdim}_M\left(X, \sigma^k(\boldsymbol{f}), d\right) \leq \mathrm{mdim}_M(X, f, d). \tag{5.1}$$

Consequently,

$$\mathrm{mdim}_M(X, \boldsymbol{f}, d)^* \leq \mathrm{mdim}_M(X, f, d).$$

Proof See the proof of Theorems 2.8 and use 4.6. □

We can prove, as in Example 2.9, that the inequality above can be strict.

Theorem 5.5 and Proposition 5.4 imply that:

Corollary 5.6 *If $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ converges uniformly to a continuous map on X, then*

$$\overline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) \leq \overline{\dim_B}(X, d) \quad \textit{and} \quad \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d) \leq \underline{\dim_B}(X, d).$$

and therefore

$$\overline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d)^* \leq \overline{\dim_B}(X, d) \quad \textit{and} \quad \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}, d)^* \leq \underline{\dim_B}(X, d).$$

In particular, if $X = [0, 1]$, then $\overline{mdim_M}(X, \boldsymbol{f}, d)^* \leq 1$.

Example 3.6 proves that the box dimension is not an upper bound for the metric mean dimension of sequences that are not convergent. Next example shows the inequality in Corollary 5.6 can be strict.

Example 5.7 For each $n \geq 1$, take $m_n = n$ and

$$f_n(x) = \begin{cases} \phi(x), & \text{if } x \in [0, a_{n+1}], \\ a_{n+1}, & \text{if } x \in [a_{n+1}, 1], \end{cases}$$

where ϕ is the map in Example 3.4. Thus f_n converges uniformly to ϕ as $n \to \infty$. In [8], Figure 3, is proved that the topological entropy $h_{top}\left((f_{n+k})_{n=1}^{\infty}\right) = k \log 3$ for each $k \geq 1$. Hence, $\overline{\mathrm{mdim}_M}([0, 1], (f_{n+k})_{n=1}^{\infty}, |\cdot|) = 0$ and therefore

$$\overline{\mathrm{mdim}_M}([0, 1], (f_n)_{n=1}^{\infty}, |\cdot|)^* = 0 < \overline{\mathrm{mdim}_M}([0, 1], \phi, |\cdot|) = 1.$$

Example 5.8 The sequence

$$g_n(x) = \begin{cases} \phi(x), & \text{if } x \in [0, a_{n+1}], \\ x, & \text{if } x \in [a_{n+1}, 1]. \end{cases}$$

converges uniformly to ϕ as $n \to \infty$, where ϕ is the map in Example 3.4. Note that $g_1^{(n+k)}|J_n = \phi^k|_{J_n}$, for $n \geq 1, k \geq 1$ (see Example 3.4). Hence

$$\text{sep}\left(2n+k, (g_i)_{i=1}^{\infty}, \varepsilon_n\right) \geq (3^{m_n}/2)^k, \quad \text{and then} \quad \text{sep}\left((g_i)_{i=1}^{\infty}, \varepsilon_n\right) \geq \log(3^{m_n}/2).$$

Therefore $\overline{\text{mdim}}_M([0,1], (g_i)_{i=1}^{\infty}, |\cdot|) \geq 1$. By (5.1) we obtain that $\overline{\text{mdim}}_M([0,1], (g_i)_{i=1}^{\infty}, |\cdot|) = 1$. Note that $\overline{\text{mdim}}_M([0,1], g_i, |\cdot|) = 0$ for any $i \geq 1$.

6 Uniform Equiconjugacy and Metric Mean Dimension

We say that the systems $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ on (X, d) and $\boldsymbol{g} = (g_n)_{n=1}^{\infty}$ on $\left(Y, d'\right)$ are *uniformly equiconjugate* if there exists a equicontinuous sequence of homeomorphisms $h_n : X \to Y$ so that $h_{n+1} \circ f_n = g_n \circ h_n$, for all $n \in \mathbb{N}$, that is, the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f_1} & X & \xrightarrow{f_2} & \dots & \xrightarrow{f_n} & X \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_{n+1}} \\ Y & \xrightarrow{g_1} & Y & \xrightarrow{g_2} & \dots & \xrightarrow{g_n} & Y \end{array}$$

is commutative for all $n \in \mathbb{N}$. In the case where $h_n = h$, for all $n \in \mathbb{N}$, we say that $\boldsymbol{f}$ and $\boldsymbol{g}$ are *uniformly conjugate*.

Note that the notion of uniform equiconjugacy does not depend on the metric on X and Y. Indeed, if d^* and $d^\star$ are another metrics on X and Y, respectively, then $(X, \boldsymbol{f}, d)$ and $(X, \boldsymbol{f}, d^*)$ are uniformly equiconjugate by the sequence $(I_X)_{n=1}^{\infty}$ and $(Y, \boldsymbol{g}, d')$ and $(Y, \boldsymbol{g}, d^\star)$ are uniformly equiconjugate by the sequence $(I_Y)_{n=1}^{\infty}$. Hence, if $(X, \boldsymbol{f}, d)$ and $(Y, \boldsymbol{g}, d')$ are uniformly equiconjugate by the sequence $(h_n)_{n=1}^{\infty}$, then $(X, \boldsymbol{f}, d^*)$ and $(Y, \boldsymbol{g}, d^\star)$ are uniformly equiconjugate by the sequence $(I_Y \circ h_n \circ I_X)_{n=1}^{\infty}$.

Theorem 6.1 *Let $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ and $\boldsymbol{g} = (g_n)_{n=1}^{\infty}$ be two non-autonomous dynamical systems defined on the metric spaces (X, d) and $\left(Y, d'\right)$ respectively.*

(i) *If $\boldsymbol{f}$ and $\boldsymbol{g}$ are uniformly conjugate then*

$$\text{mdim}(X, \boldsymbol{f}) = \text{mdim}(X, \boldsymbol{g}).$$

(ii) *If $(X, \boldsymbol{f})$ and $(Y, \boldsymbol{g})$ are uniformly equiconjugate by a sequence of homeomorphisms $(h_n)_{n=1}^{\infty}$ that satisfies $\inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right)\right\} > 0$ for any $y_1, y_2 \in Y$, then (see (3.1))*

$$\text{mdim}_M(X, \boldsymbol{f}) \geq \text{mdim}_M(Y, \boldsymbol{g}).$$

(iii) *If $(X, \boldsymbol{f})$ and $(Y, \boldsymbol{g})$ are uniformly equiconjugate by a sequence of homeomorphisms $(h_n)_{n=1}^{\infty}$ that satisfies $\inf_n \left\{d'(h_n(x_1), h_n(x_2))\right\} > 0$ for any $x_1, x_2 \in X$, then*

$$\text{mdim}_M(X, \boldsymbol{f}) \leq \text{mdim}_M(Y, \boldsymbol{g}).$$

(iv) *If $(X, \boldsymbol{f})$ and $(Y, \boldsymbol{g})$ are uniformly equiconjugate by a sequence of homeomorphisms $(h_n)_{n=1}^{\infty}$ that satisfies $\inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right), d'(h_n(x_1), h_n(x_2))\right\} > 0$ for any*

$y_1, y_2 \in Y$ and $x_1, x_2 \in X$, then

$$\mathrm{mdim}_M(X,\boldsymbol{f}) = \mathrm{mdim}_M(Y,\boldsymbol{g}).$$

Proof (i) Let $h : X \to Y$ be a homeomorphism which conjugates $\boldsymbol{f}$ and $\boldsymbol{g}$, i.e., $h \circ f_1^{(n)} = g_1^{(n)} \circ h$ for all $n \in \mathbb{N}$. For an open cover α of X, consider $\beta = h(\alpha)$, which is an open cover of Y. Now we notice that

$$\begin{aligned}\beta_0^{n-1} &= h(\alpha) \vee g_1^{-1}(h(\alpha)) \vee \cdots \vee \left(g_1^{(n-1)}\right)^{-1}(h(\alpha)) \\ &= h(\alpha) \vee \left(h \circ f_1^{-1} \circ h^{-1}\right)(h(\alpha)) \vee \cdots \vee \left(h \circ \left(f_1^{(n-1)}\right)^{-1} \circ h^{-1}\right)(h(\alpha)) \\ &= h\left(\alpha_0^{n-1}\right).\end{aligned}$$

It implies that $\mathcal{D}\left(h\left(\alpha_0^{n-1}\right)\right) = \mathcal{D}\left(\alpha_0^{n-1}\right)$. Since, for any open cover β of Y is of the form $h(\alpha)$, for some open cover α of X,

$$\mathrm{mdim}(X,\boldsymbol{f}) = \sup_{\alpha} \lim_{n\to\infty} \frac{\mathcal{D}\left(\alpha_0^{n-1}\right)}{n} = \sup_{\beta} \lim_{n\to\infty} \frac{\mathcal{D}\left(\beta_0^{n-1}\right)}{n} = \mathrm{mdim}(Y,\boldsymbol{g}).$$

(ii) Let $(h_n)_{n=1}^{\infty}$ be the sequence of equicontinuous homeomorphisms that equiconjugates $\boldsymbol{f}$ and $\boldsymbol{g}$. So,

$$f_n \circ \cdots \circ f_1 = h_{n+1}^{-1} \circ g_n \circ \cdots \circ g_1 \circ h_1.$$

By assumption we have

$$\inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right)\right\} > 0, \text{ for any } y_1 \neq y_2 \in Y.$$

Hence, we can define on Y the metric

$$d^\star(y_1, y_2) := \inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right)\right\}.$$

In particular, if $S \subset X$ is a $(m,\boldsymbol{f},\varepsilon)$-spanning set of X in the metric d and $x_1, x_2 \in S$, then

$$\begin{aligned} d_m^\star(h_1(x_1), h_1(x_2)) &= \max\left\{d^\star(h_1(x_1), h_1(x_2)), \ldots, d^\star\left(g_1^{m-1}(h_1(x_1)), g_1^{m-1}(h_1(x_2))\right)\right\} \\ &\leq \max\left\{d(x_1, x_2), d\left(h_2^{-1}(g_1(h_1(x_1))), h_2^{-1}(g_1(h_1(x_2)))\right),\right. \\ &\quad \left. \ldots, d\left(h_{m+1}^{-1}\left(g_1^{m-1}(h_1(x_1))\right), h_{m+1}^{-1}\left(g_1^{m-1}(h_1(x_2))\right)\right)\right\} \\ &= d_m(x_1, x_2) \leq \varepsilon. \end{aligned}$$

It follows that $h_1(S)$ is an $(m, \boldsymbol{g}, \varepsilon)$-spanning set of Y in the metric $d^\star$. So we obtain that

$$\mathrm{mdim}_M(X,\boldsymbol{f}, d) \geq \mathrm{mdim}_M\left(Y, \boldsymbol{g}, d^\star\right),$$

and therefore $\mathrm{mdim}_M(X,\boldsymbol{f}) \geq \mathrm{mdim}_M(Y,\boldsymbol{g})$.

By an analogous argument we can prove (iii). Item (iv) follows from (ii) and (iii). □

Clearly the theorem implies that if $\phi : X \to X$ and $\psi : X \to X$ are topologically conjugate continuous maps, then

$$\mathrm{mdim}_M(X, \phi) = \mathrm{mdim}_M(X, \psi),$$

which is a well-known fact.

The next corollaries follow from Theorem 6.1.

Corollary 6.2 *If* $f_1, \dots, f_i, g_1, \dots, g_i$ *are homeomorphisms,* $\boldsymbol{f} = (f_1, \dots, f_i, f_{i+1}, f_{i+2}, \dots)$ *and* $\boldsymbol{g} = (g_1, \dots, g_i, f_{i+1}, f_{i+2}, \dots)$*, then*

$$\text{mdim}_M(X,\boldsymbol{f}) = \text{mdim}_M(Y, \boldsymbol{g}).$$

Proof Note that the following diagram is commutative

$$\begin{array}{ccccccccccc}
X & \xrightarrow{f_1} & X & \xrightarrow{f_i} & \dots X & \xrightarrow{f_i} & X & \xrightarrow{f_{i+1}} & X & \xrightarrow{f_{i+2}} & X \\
{\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle Id_X} \\
X & \xrightarrow{g_1} & X & \xrightarrow{g_i} & \dots X & \xrightarrow{g_i} & X & \xrightarrow{f_{i+1}} & X & \xrightarrow{f_{i+2}} & X
\end{array}$$

where I_X is the identity of X and $h_i = g_i^{-1} \circ f_i$, $h_{i-1} = g_{i-1}^{-1} \circ h_i \circ f_{i-1}, \dots,$ $h_1 = g_1^{-1} h_2 f_1$. Furthermore, $(h_1, h_2, \dots, h_i, I_X, I_X, \dots)$ is an equicontinuous sequence of homeomorphisms. Therefore, $\boldsymbol{f}$ and $\boldsymbol{g}$ are uniformly equiconjugate. The corollary follows from Theorem 6.1, since the infimum $\inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right), d(h_n(x_1), h_n(x_2))\right\} > 0$ is taken over a finite set. □

Next corollary means that if $\boldsymbol{f}$ is a sequence of homeomorphisms then the metric mean dimension is independent on the firsts elements in the sequence $\boldsymbol{f}$.

Corollary 6.3 *Let* $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ *be a non-autonomous dynamical system consisting of homeomorphisms. For any* $i, j \in \mathbb{N}$ *we have*

$$\text{mdim}_M\left(X, \sigma^i(\boldsymbol{f})\right) = \text{mdim}_M\left(X, \sigma^j(\boldsymbol{f})\right).$$

Proof It is sufficient to prove that $\text{mdim}_M\left(X, \sigma^i(\boldsymbol{f})\right) = \text{mdim}_M(X,\boldsymbol{f})$ for all $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$. Take $\boldsymbol{g} = (g_n)_{n\in\mathbb{N}}$, where, for each $n \leq i$, $g_n = I$ is the identity on X and $g_n = f_n$ for $n > i$. It follows from Corollary 6.2 that

$$\text{mdim}_M(X,\boldsymbol{f}) = \text{mdim}_M(X, \boldsymbol{g}).$$

For each $x, y \in X$ and $n > i$ we have

$$\begin{aligned}
&\max\left\{d(x, y), \dots, d\left(g_1^{(i-1)}(x), g_1^{(i-1)}(y)\right), \dots, d\left(g_1^{(n-1)}(x), g_1^{(n-1)}(y)\right)\right\} \\
&= \max\left\{d(x, y), d(g_i(x), g_i(y)), \dots, d\left(g_i^{(n-i)}(x), g_i^{(n-i)}(y)\right)\right\} \\
&= \max\left\{d(x, y), d(f_i(x), f_i(y)), \dots, d\left(f_i^{(n-i)}(x), f_i^{(n-i)}(y)\right)\right\}.
\end{aligned}$$

Hence

$$\text{mdim}_M(X,\boldsymbol{f}) = \text{mdim}_M(X, \boldsymbol{g}) = \text{mdim}_M\left(X, \sigma^i(\boldsymbol{f})\right),$$

which proves the corollary. □

Next corollary follows from Corollary 6.3 and Proposition 4.1 (see the proof of Corollary 2.7).

Corollary 6.4 *For any homeomorphisms* f *and* g *defined on* X*, we have*

$$\text{mdim}_M(X, f \circ g) = \text{mdim}_M(X, g \circ f).$$

7 On the Continuity of the Metric Mean Dimension

In this section we will show some results related to the continuity of the metric mean dimension of sequences of diffeomorphisms defined on a manifold. For any $r \geq 0$, set

$$\mathcal{C}^r(X) = \left\{(f_n)_{n=1}^{\infty} : f_n : X \to X \text{ is a } C^r\text{-map}\right\} = \prod_{i=1}^{+\infty} \mathrm{C}^r(X),$$

where $\mathrm{C}^r(X) = \{\phi : X \to X : \phi \text{ is a } C^r\text{-map}\}$.[1] Hence $\mathcal{C}^r(X)$ can be endowed with the *product topology*, which is generated by the sets

$$\mathcal{U} = \prod_{i=1}^{j} \mathrm{C}^r(X) \times \prod_{i=j+1}^{j+m} U_i \times \prod_{i>j+m}^{+\infty} \mathrm{C}^r(X),$$

where U_i is an open subset of $\mathrm{C}^r(X)$, for $j+1 \leq i \leq j+m$, for some $j, m \in \mathbb{N}$. The space $\mathcal{C}^r(X)$ with the product topology will be denoted by $\left(\mathcal{C}^r(X), \tau_{prod}\right)$. We can consider the map

$$\begin{aligned} \underline{\mathrm{mdim}}_M : (\mathcal{C}^r(X), \tau_{prod}) &\to \mathbb{R} \cup \{+\infty\} \\ \boldsymbol{f} &\to \underline{\mathrm{mdim}}_M(\boldsymbol{f}, X). \end{aligned}$$

Clearly, if $\underline{\mathrm{mdim}}_M$ is a constant map, then is continuous.

Proposition 7.1 *If* $\underline{mdim}_M : (\mathcal{C}^r(X), \tau_{prod}) \to \mathbb{R} \cup \{+\infty\}$ *is not constant then is discontinuous at any* $\boldsymbol{f} \in \mathcal{C}^r(X)$.

Proof Fix $\boldsymbol{f} = (f_n)_{n=1}^{\infty} \in \mathcal{C}^r(X)$. Since $\underline{\mathrm{mdim}}_M$ is not constant, there exists $\boldsymbol{g} = (g_n)_{n=1}^{\infty} \in \mathcal{C}^r(X)$ such that $\underline{\mathrm{mdim}}_M(X, \boldsymbol{g}) \neq \underline{\mathrm{mdim}}_M(X, \boldsymbol{f})$. Let $\mathcal{V} \in \tau_{prod}$ be any open neighborhood of $\boldsymbol{f}$. For some $k \in \mathbb{N}$, the sequence $\boldsymbol{j} = (j_n)_{n=1}^{\infty}$, defined by

$$j_n = \begin{cases} f_n & \text{if } n = 1, \ldots, k \\ g_n & \text{if } n > k, \end{cases}$$

belongs to $\mathcal{V}$, by definition of τ_{prod}. It is follow from Corollary 6.2 that $\underline{\mathrm{mdim}}_M(X, \boldsymbol{j}) = \underline{\mathrm{mdim}}_M(X, \boldsymbol{g})$. which proves the proposition. □

Let $d^1(\cdot, \cdot)$ be a C^1-metric on $\mathrm{C}^1(X)$. Suppose that $\sup_{n \in \mathbb{N}} \|Df_n\| < \infty$. For any $K > 0$, if $d^1(g_n, f_n) < K$, then $\sup_{n \in \mathbb{N}} \|Dg_n\| < \infty$ and therefore $\underline{\mathrm{mdim}}_M(X, \boldsymbol{g}, d) = 0$. On the other hand, if $\sup_{n \in \mathbb{N}} \|Df_n\| = \infty$, then $\underline{\mathrm{mdim}}_M(X, \boldsymbol{f}, d)$ is not necessarily zero.

In [15], Section 6, is proved that:

Proposition 7.2 *If* $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ *is a sequence of* C^1*-diffeomorphisms, there exists a sequence of positive numbers* $(\delta_n)_{n=1}^{\infty}$ *such that every sequence of diffeomorphisms* $\boldsymbol{g} = (g_n)_{n=1}^{\infty}$ *with* $d^1(f_n, g_n) < \delta_n$ *for each* $n \geq 1$, *is uniformly equiconjugate to* $\boldsymbol{f}$ *by a sequence* $(h_n)_{n=1}^{\infty}$ *such that* $h_n \to I_X$ *as* $n \to \infty$.

Note that, if $h_n \to I_X$ as $n \to \infty$, then for any $x_1 \neq x_2 \in X$ and $y_1 \neq y_2 \in Y$ we have $\inf_n \left\{d\left(h_n^{-1}(y_1), h_n^{-1}(y_2)\right), d(h_n(x_1), h_n(x_2))\right\} > 0$. Hence, it follows from Theorems 6.1 and Proposition 7.2 that

[1] If $r \geq 1$ we assume that X is a Riemannian manifold

Corollary 7.3 *Given a sequence of diffeomorphisms* $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$, *there exists a sequence of positive numbers* $(\delta_n)_{n=1}^{\infty}$ *such that if* $\boldsymbol{g} = (g_n)_{n=1}^{\infty}$ *is a sequence of diffeomorphisms such that* $d^1(f_n, g_n) < \delta_n$ *for each* $n \geq 1$, *then*

$$\underline{\mathrm{mdim}_M}(X, \boldsymbol{g}) = \underline{\mathrm{mdim}_M}(X, \boldsymbol{f}).$$

Roughly, Corollary 7.3 means that if $d^1(f_n, g_n)$ converges very quickly to zero as $n \to \infty$, then

$$\underline{\mathrm{mdim}_M}(X, \boldsymbol{f}) = \underline{\mathrm{mdim}_M}(X, \boldsymbol{g}).$$

For each sequence of diffeomorphisms $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ and a sequence of positive numbers $\varepsilon = (\varepsilon_n)_{n=1}^{\infty}$, a *strong basic neighborhood* of $\boldsymbol{f}$ is the set

$$B^r(\boldsymbol{f}, \varepsilon) = \left\{\boldsymbol{g} = (g_n)_{n=1}^{\infty} : g_n \text{ is a } C^r\text{-diffeomorphism and } d(f_n, g_n) < \varepsilon_n, \text{ for all } n \in \mathbb{N}\right\}.$$

The *strong topology* (or *Whitney topology*) on $\mathcal{C}^r(X)$ is generated by the strong basic neighborhoods of each $\boldsymbol{f} \in \mathcal{C}^r(X)$. The space $\mathcal{C}^r(X)$ with the strong topology will be denoted by $(\mathcal{C}^r(X), \tau_{str})$.

Corollary 7.4 *For* $r \geq 1$, *let* $\mathcal{D}^r(X) \subseteq \mathcal{C}^r(X)$ *be the set consisting of diffeomorphisms. Then*

$$\underline{\mathrm{mdim}_M} : (\mathcal{D}^r(X), \tau_{str}) \to \mathbb{R} \cup \{+\infty\}$$

is a continuous map.

Proof Let $\boldsymbol{f} \in \mathcal{D}^r(X)$. If follows from Theorem 7.2 that there exists a strong basic neighborhood $B^r(\boldsymbol{f}, (\delta_n)_{n=1}^{\infty})$ such that every $\boldsymbol{g} \in B^r(\boldsymbol{f}, (\delta_n)_{n=1}^{\infty})$ is uniformly equiconjugate to $\boldsymbol{f}$. Thus, from Proposition 6.1 we have $\underline{\mathrm{mdim}_M}(X, \boldsymbol{g}) = \underline{\mathrm{mdim}_M}(X, \boldsymbol{f})$ for all $\boldsymbol{g} \in B^r(\boldsymbol{f}, (\delta_n)_{n=1}^{\infty})$, which proves the corollary. □

A real valued function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is called *lower* (respectively *upper*) *semi-continuous on a point* $x \in X$ if

$$\liminf_{y \to x} \varphi(y) \geq \varphi(x) \qquad \left(\text{respectively} \limsup_{y \to x} \varphi(y) \leq \varphi(x)\right).$$

φ is called *lower* (respectively *upper*) *semi-continuous* if is lower (respectively upper) semi-continuous on any point of X.

Remark 7.5 From now on, we will consider $\tilde{X} = [0, 1]$ or $\mathbb{S}^1$.

Misiurewicz in [14], Corollary 1, proved that $h_{top} : C^0([0, 1]) \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous. For the case of the metric mean dimension we have:

Proposition 7.6 *mdim*$_M$: $C^0\left(\tilde{X}\right) \to \mathbb{R}$ *is nor lower neither upper semi-continuous on maps with metric mean dimension in* $(0, 1)$. *Furthermore, mdim*$_M$: $C^0\left(\tilde{X}\right) \to \mathbb{R}$ *is not lower semi-continuous on maps with metric mean dimension in* $(0, 1]$ *and is not upper semi-continuous on maps with metric mean dimension in* $[0, 1)$.

Proof Let φ be a continuous map on $\tilde{X}$. If $\mathrm{mdim}_M(\varphi) = 1$, we can approximate φ by a continuous map with zero metric mean dimension (take a sequence of C^1-maps converging to φ). Next, suppose that $\mathrm{mdim}_M(\varphi) = 0$. Firstly, take $\tilde{X} = [0, 1]$. Fix $\varepsilon > 0$. Let p^* be a fixed point of φ. Choose $\delta > 0$ such that $d(\varphi(x), \varphi(p^*)) < \varepsilon/2$ for any x with $d(x, p^*) < \delta$. Let ϕ and T_2 be as in Example 3.4, with $J_1 = [0, p^*]$, $J_2 = [p^*, p^* + \delta/2]$, $J_3 = [p^* + \delta/2, p^* + \delta]$ and $J_4 = [p^* + \delta, 1]$. Take the continuous map ψ on $[0, 1]$ defined as

$$\psi(x) = \begin{cases} \varphi(x), & \text{if } x \in J_1 \cup J_4, \\ T_2^{-1}\phi T_2(x), & \text{if } x \in J_2, \\ \psi_1(x), & \text{if } x \in J_3, \end{cases}$$

where ψ_1 is the affine map on J_3 such that $\psi_1(p^* + \delta/2) = (p^* + \delta/2)$ and $\psi_1(p^* + \delta) = \varphi(p^* + \delta)$. Note that $d(\psi, \varphi) < \varepsilon$. It follows from Proposition 4.3 that

$$\mathrm{mdim}_M\left(\tilde{X}, \psi, |\cdot|\right) = \max\left\{\mathrm{mdim}_M\left(\tilde{X}, \psi|_{J_1\cup J_3\cup J_4}, |\cdot|\right), \mathrm{mdim}_M\left(\tilde{X}, \psi|_{J_2}, |\cdot|\right)\right\} = 1,$$

since $\mathrm{mdim}_M\left(\tilde{X}, \psi|_{J_1\cup J_3\cup J_4}, |\cdot|\right) \leq \mathrm{mdim}_M\left(\tilde{X}, \varphi, |\cdot|\right) = 0$. Analogously we can prove that any $\varphi \in C^0([0, 1])$ with metric mean dimension in $(0, 1)$ can be approximated by both a continuous map with metric mean dimension equal to 1 and a continuous map with metric mean dimension equal to 0. These facts prove the proposition for $\tilde{X} = [0, 1]$. For $\tilde{X} = \mathbb{S}^1$, we can approximate any $\varphi \in C^0\left(\mathbb{S}^1\right)$ by a map φ^* with periodic points. We can prove analogously that φ^* can be approximate by a continuous map on $\mathbb{S}^1$ with metric mean dimension equal to 0 or equal to 1, which proves the proposition for $\tilde{X} = \mathbb{S}^1$. □

Next, Kolyada and Snoha in [8], Theorem F, showed that $h_{top} : \mathcal{C}([0, 1]) \to \mathbb{R} \cup \{\infty\}$ is not lower semi-continuous, endowing $\mathcal{C}([0, 1])$ with the metric

$$D\left((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}\right) = \sup_{n\in\mathbb{N}} \max_{x\in[0,1]} |f_n(x) - g_n(x)|.$$

Furthermore, they proved in Theorem G that $h_{top} : \mathcal{C}([0, 1]) \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous on any constant sequence $(\phi, \phi, \ldots) \in \mathcal{C}\left(\tilde{X}\right)$. However, It follows from Proposition 7.6 that:

Corollary 7.7 *$\mathrm{mdim}_M : \mathcal{C}\left(\tilde{X}\right) \to \mathbb{R}$ is nor lower neither upper semi-continuous on any constant sequence $(\phi, \phi, \ldots) \in \mathcal{C}\left(\tilde{X}\right)$. Consequently, $\mathrm{mdim}_M : \mathcal{C}\left(\tilde{X}\right) \to \mathbb{R} \cup \{\infty\}$ is nor lower neither upper semi-continuous.*

Take $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ on $\tilde{X}$ defined by $f_n = \psi^{2^n}$ for each $n \in \mathbb{N}$, where ψ is the map from Example 3.4. We have $\mathrm{mdim}_M(\tilde{X}, \boldsymbol{f}, |\cdot|) = \infty$ (see Example 3.8). Thus there exist non-autonomous dynamical systems on $\tilde{X}$ with infinite metric mean dimension. Consequently $\mathrm{mdim}_M : \mathcal{C}\left(\tilde{X}\right) \to \mathbb{R} \cup \{\infty\}$ is unbounded.

We finish this work with the next result:

Theorem 7.8 *$\mathrm{mdim}_M : \mathcal{C}\left(\tilde{X}\right) \to \mathbb{R} \cup \{\infty\}$ is not lower semi-continuous on any non-autonomous dynamical system with non-zero metric mean dimension.*

Proof Let $\boldsymbol{f} = (f_n)_{n=1}^{\infty}$ be a non-autonomous dynamical system with positive metric mean dimension. Let λ_m be a sequence in $[0, 1]$ such that $\lambda_m \to 1$ and $\lambda_m \cdots \lambda_1 \to 0$ as $m \to \infty$. Take $\boldsymbol{g}_m = (\lambda_{m+n} f_n)_{n=1}^{\infty}$. Thus $\boldsymbol{g}_m \to \boldsymbol{f}$ as $m \to \infty$. However, for any $x \in \tilde{X}$,

$(g_m)^{(k)}(x) \to 0$ as $k \to \infty$. Consequently, the metric mean dimension of $\boldsymbol{g}_m$ is zero for each $m \in \mathbb{N}$. $\square$

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