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Hölder continuous maps on the interval with positive metric mean dimension

Funciones Hölder continuas en el intervalo con dimensión métrica media positiva

Jeovanny de Jesus Muentes Acevedo[1],
Sergio Augusto Romaña Ibarra[2],
Raibel de Jesus Arias Cantillo[3]

[1]Universidad Tecnológica de Bolívar, Cartagena, Colombia

[2]Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil

[3]Universidade Federal de Maranhão, San Luis, Brasil

Abstract. Fix a compact metric space X with finite topological dimension. Let $C^0(X)$ be the space of continuous maps on X and $H^\alpha(X)$ the space of α-Hölder continuous maps on X, for $\alpha \in (0,1]$. Let $H^1(X)$ be the space of Lipschitz continuous maps on X. We have

$$H^1(X) \subset H^\beta(X) \subset H^\alpha(X) \subset C^0(X), \quad \text{where } 0 < \alpha < \beta < 1.$$

It is well-known that if $\phi \in H^1(X)$, then ϕ has metric mean dimension equal to zero. On the other hand, if X is a manifold, then $C^0(X)$ contains a residual subset whose elements have positive metric mean dimension. In this work we will prove that, for any $\alpha \in (0,1)$, there exists $\phi \in H^\alpha([0,1])$ with positive metric mean dimension.

Key words and phrases. metric mean dimension, topological entropy, Hölder continuous maps.

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Resumen. Fijemos un espacio métrico compacto X con dimensión topológica finita. Sea $C^0(X)$ el espacio de funciones continuas en X y $H^\alpha(X)$ el espacio de funciones α-Hölder continuas en X, para $\alpha \in (0,1]$. Sea $H^1(X)$ el espacio de funciones Lipschitz continuas en X. Tenemos la siguiente inclusión:

$$H^1(X) \subset H^\beta(X) \subset H^\alpha(X) \subset C^0(X), \quad \text{donde } 0 < \alpha < \beta < 1.$$

Es bien sabido que si $\phi \in H^1(X)$, entonces ϕ tiene dimensión métrica media igual a cero. Por otro lado, si X es una variedad Riemanniana compacta, entonces $C^0(X)$ contiene un subconjunto residual cuyos elementos tienen dimensión métrica media positiva. En este trabajo demostraremos que, para cualquier $\alpha \in (0,1)$, existe $\phi \in H^{\alpha}([0,1])$ con dimensión métrica media positiva.

Palabras y frases clave. dimensión métrica media, entropía topológica, funciones Hölder continuas.

1. Introduction

In [6], Lindestrauss and Weiss introduced the notion of metric mean dimension for any continuous map $\phi : X \to X$, where X is a compact metric space with metric d. We will denote by $\underline{\text{mdim}}_{\text{M}}(X, d, \phi)$ and $\overline{\text{mdim}}_{\text{M}}(X, d, \phi)$, respectively, the lower and upper metric mean dimension of $\phi : X \to X$. We have

$$\underline{\text{mdim}}_{\text{M}}(X, d, \phi) \leq \overline{\text{mdim}}_{\text{M}}(X, d, \phi). \tag{1}$$

Denote by $h_{\text{top}}(\phi)$ the topological entropy of $\phi : X \to X$. We have if $\underline{\text{mdim}}_{\text{M}}(X, d, \phi) > 0$, then $h_{\text{top}}(\phi) = \infty$. Example 4.3 proves that there exist continuous maps $\phi : X \to X$ with infinite topological entropy and metric mean dimension equal to zero.

Let N be a compact Riemannian manifold with topological dimension $\dim(N) = n$. In [2], the authors prove that, if $n \geq 2$, then the set consisting of all homeomorphism on N with upper metric mean dimension equal to n is residual in $\text{Hom}(N)$. This fact is proved in [1] for $C^0(N)$ instead of $\text{Hom}(N)$. On the other hand, any Lipschitz continuous map defined on a finite dimensional compact metric space has finite entropy (see [4], Theorem 3.2.9), therefore, it has metric mean dimension equal to zero.

For any $\alpha \in (0,1)$, we denote by $H^{\alpha}([0,1])$ the set consisting of α-Hölder continuous maps on the interval $[0,1]$. Hazard in [3] proves that there exist continuous maps on the interval with infinite entropy which are α-Hölder for any $\alpha \in (0,1)$. However, the example constructed by Hazard has zero metric mean dimension (see Example 4.3). In this work, we will show that, for any $\alpha \in (0,1)$, there exists a $\phi \in H^{\alpha}([0,1])$ with

$$\underline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi) = \overline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi) = 1 - \alpha.$$

In the next section, we will present the definition of metric mean dimension. In Section 3, we will show some results about the metric mean dimension for continuous maps with horseshoes and several examples. Although the inequality given in (1) is clear, we do not know any reference of an explicit example of a continuous map on the interval for which the inequality is strict. We will construct this kind of examples in Section 3. Furthermore, we prove that for

$a, b \in [0,1]$, with $a < b$, the set consisting of continuous maps $\phi : [0,1] \to [0,1]$ such that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = a \quad \text{and} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = b$$

is dense in $C^0([0,1])$. In Section 4 we show the existence of Hölder continuous maps with positive metric mean dimension. Finally, in the last section we state some conjectures that arise from this research.

2. Metric mean dimension for continuous maps

Throughout this work, X will be a compact metric space endowed with a metric d and $\phi : X \to X$ a continuous map. For any $n \in \mathbb{N}$, we define the metric $d_n : X \times X \to [0, \infty)$ by

$$d_n(x, y) = \max\{d(x, y), d(\phi(x), \phi(y)), \ldots, d(\phi^{n-1}(x), \phi^{n-1}(y))\}.$$

Definition 2.1. Fix $\varepsilon > 0$.

- We say that $A \subset X$ is an (n, ϕ, ε)-*separated* set if $d_n(x, y) > \varepsilon$, for any two distinct points $x, y \in A$. We denote by $sep(n, \phi, \varepsilon)$ the maximal cardinality of any (n, ϕ, ε)-separated subset of X.
- We say that $E \subset X$ is an (n, ϕ, ε)-*spanning* set for X if for any $x \in X$ there exists $y \in E$ such that $d_n(x, y) < \varepsilon$. Let $span(n, \phi, \varepsilon)$ be the minimum cardinality of any (n, ϕ, ε)-spanning subset of X.

Definition 2.2. The *topological entropy* of (X, ϕ, d) is defined by

$$h_{top}(\phi) = \lim_{\varepsilon \to 0} sep(\phi, \varepsilon) = \lim_{\varepsilon \to 0} span(\phi, \varepsilon),$$

where

$$sep(\phi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log sep(n, \phi, \varepsilon) \text{ and } span(\phi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log span(n, \phi, \varepsilon).$$

Definition 2.3. We define the *lower metric mean dimension* and the *upper metric mean dimension* of (X, d, ϕ) by

$$\underline{mdim}_M(X, d, \phi) = \liminf_{\varepsilon \to 0} \frac{sep(\phi, \varepsilon)}{|\log \varepsilon|} \quad \text{and} \quad \overline{mdim}_M(X, d, \phi) = \limsup_{\varepsilon \to 0} \frac{sep(\phi, \varepsilon)}{|\log \varepsilon|},$$

respectively.

We also have that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) = \liminf_{\varepsilon \to 0} \frac{\mathrm{span}(\phi, \varepsilon)}{|\log \varepsilon|} \text{ and } \overline{\mathrm{mdim}}_{\mathrm{M}}(X, d, \phi) = \limsup_{\varepsilon \to 0} \frac{\mathrm{span}(\phi, \varepsilon)}{|\log \varepsilon|}.$$

***Remark* 2.4.** Throughout the paper, by simplicity, $mdim_{\mathrm{M}}(X,d,\phi)$ will be denote both quantities $\underline{mdim}_{\mathrm{M}}(X,d,\phi)$ and $\overline{mdim}_{\mathrm{M}}(X,d,\phi)$.

***Remark* 2.5.** Topological entropy does not depend on the metric d. However, the metric mean dimension depends on the metric (see [6]), therefore, it is not an invariant under topological conjugacy. If $X=[0,1]$, we consider the metric $d(x,y)=|x-y|$, for every $x,y\in[0,1]$. We will denote this metric by $|\cdot|$.

***Remark* 2.6.** For any continuous map $\phi: X\to X$, it is proved in [9] that

$$0\leq \overline{mdim}_M(X,d,\phi)\leq \overline{dim}_B(X,d) \quad \text{and} \quad 0\leq \underline{mdim}_M(X,d,\phi)\leq \underline{dim}_B(X,d), \tag{2}$$

where $\underline{dim}_B(X,d)$ and $\overline{dim}_B(X,d)$ are respectively the lower and upper box dimension of X with respect to d.

From the remark above we have that, if N is an n-dimensional compact Riemannian manifold with Riemannian metric d, then we have that

$$0\leq \mathrm{mdim}_{\mathrm{M}}(N,d,\phi)\leq n,$$

for any continuous map $\phi: N\to N$. In particular, if $\phi:[0,1]\to[0,1]$ is a continuous map, we have that

$$0\leq \mathrm{mdim}_{\mathrm{M}}([0,1],|\cdot|,\phi)\leq 1. \tag{3}$$

3. Horseshoes and metric mean dimension

An s-*horseshoe* for $\phi:[0,1]\to[0,1]$ is an interval $J=[a,b]\subseteq[0,1]$ which has a partition into s subintervals $J_1,\ldots,J_s$, such that $J_j^\circ\cap J_i^\circ=\varnothing$ for $i\neq j$ and $J\subseteq\phi(\overline{J}_i)$ for each $i=1,\ldots,s$ (in Figure 1 we show a 3-horseshoe). The subintervals J_i will be called *legs* of the horseshoe J and the length $|J|:=b-a$ is its *size*.

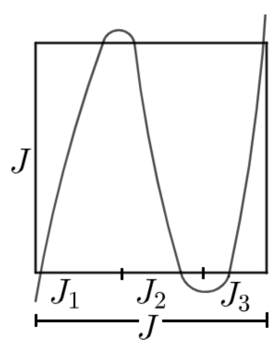


FIGURE 1. J is an 3-horseshoe

Misiurewicz in [8], proved if $\phi : [0,1] \to [0,1]$ is a continuous map with $h_{\text{top}}(\phi) > 0$, then there exist sequences of positive integers k_n and s_n such that, for each n, ϕ^{k_n} has an s_n- horseshoe and

$$h_{\text{top}}(\phi) = \lim_{n\to\infty} \frac{1}{k_n} \log s_n.$$

For metric mean dimension, in [9], Lemma 6, it is proved that:

Lemma 3.1. Suppose that $I_k = [a_{k-1}, a_k] \subseteq [0,1]$ is an s_k-horseshoe for $\phi : [0,1] \to [0,1]$ consisting of s_k subintervals with the same length $I_k^1, \dots, I_k^{s_k}$. Setting $\varepsilon_k = \frac{|I_k|}{s_k}$, we have

$$sep(\phi, \varepsilon_k) \geq \log(s_k/2).$$

The lemma above provides a lower bound for the *upper* metric mean dimension of a continuous map $\phi : [0,1] \to [0,1]$ with a sequence of horseshoes, since, with the conditions presented, we can have that

$$\overline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi) \geq \limsup_{k\to\infty} \frac{\log s_k}{|\log \varepsilon_k|} = \limsup_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}, \tag{4}$$

as can be seen in the proof of Proposition 8 from [9]. In order to obtain the exact value of the (lower and upper) metric mean dimension of a continuous map on the interval, we must be carefully choosing both the number and the size of the legs of the horseshoes. Inspired by the examples presented in [5], in [1] we prove the next result, which, together with the Lemma 3.1, will give us examples of continuous maps on the interval with metric mean dimension equal to a fixed value (see Examples 3.4, 3.5 and 4.4).

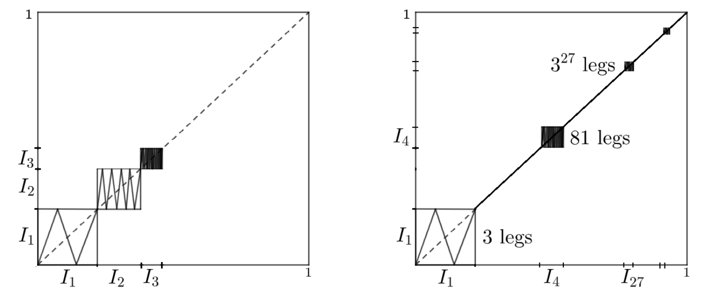
Theorem 3.2. *Suppose for each $k \in \mathbb{N}$ there exists a s_k-horseshoe for $\phi \in C^0([0,1])$, $I_k = [a_{k-1}, a_k] \subseteq [0,1]$, consisting of sub-intervals with the same length $I_k^1, I_k^2, \dots, I_k^{s_k}$ and $[0,1] = \cup_{k=1}^{\infty} I_k$. We can rearrange the intervals and suppose that $2 \leq s_k \leq s_{k+1}$ for each k. If each $\phi|_{I_k^i} : I_k^i \to I_k$ is a bijective affine map for all k and $i = 1, \dots, s_k$, we have*

i. $\underline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi) \leq \liminf_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}$.

ii. If the limit $\lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}$ *exists, then* $\overline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi) = \lim_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}$.

In Figure 3 we present the graph of a continuous map $\phi : [0,1] \to [0,1]$ such that each I_k is an 3^k-horseshoe for ϕ. Note that in Theorem 3.2,i. is presented an upper bound for the *lower* metric mean dimension and in ii. a condition to obtain the exact value of the *upper* metric mean dimension for a certain class of continuous maps on the interval.

In the next proposition we show a lower bound for the *lower* metric mean dimension of a continuous map on the interval satisfying the conditions in Theorem 3.2.



(a) Each I_k is an 3^k-horseshoe (b) Each I_k is an 3^k-horseshoe

Proposition 3.3. *If $\phi : [0,1] \to [0,1]$ satisfies the conditions in Theorem 3.2, we have*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) \geq \liminf_{k\to\infty} \frac{\frac{\log s_{k-1}}{\log s_k}}{1 - \frac{\log |I_k|}{\log s_k}}. \tag{5}$$

If $\liminf_{k\to\infty} \frac{\frac{\log s_{k-1}}{\log s_k}}{1-\frac{\log |I_k|}{\log s_k}} = \liminf_{k\to\infty} \frac{1}{\left|1-\frac{\log |I_k|}{\log s_k}\right|}$, *we have*

$$\underline{mdim}_M([0,1], |\cdot|, \phi) = \liminf_{k\to\infty} \frac{1}{\left|1 - \frac{\log |I_k|}{\log s_k}\right|}.$$

Proof. Take any $\varepsilon \in (0,1)$. For any $k \geq 1$ set $\varepsilon_k = \frac{|I_k|}{s_k}$. There exists some $k \geq 1$ such that $\varepsilon \in [\varepsilon_k, \varepsilon_{k-1}]$. We have

$$\mathrm{sep}(n, \phi, \varepsilon) \geq \mathrm{sep}(n, \phi, \varepsilon_{k-1}) \geq \mathrm{sep}(n, \phi|_{I_{k-1}}, \varepsilon_{k-1}), \quad \text{for any } n \geq 1,$$

and thus $\mathrm{sep}(\phi, \varepsilon) \geq \mathrm{sep}(\phi|_{I_{k-1}}, \varepsilon_{k-1})$. Hence, from Lemma 3.1, it follows that $\mathrm{sep}(\phi,\varepsilon) \geq \log\left(\frac{s_{k-1}}{2}\right)$. Therefore,

$$\begin{aligned}\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) &= \liminf_{\varepsilon\to 0} \frac{\mathrm{sep}(\phi,\varepsilon)}{|\log \varepsilon|} \geq \liminf_{k\to\infty} \frac{\log s_{k-1}}{|\log \varepsilon_k|} \\ &= \liminf_{k\to\infty} \frac{\log s_{k-1}}{\log s_k - \log |I_k|} = \liminf_{k\to\infty} \frac{\frac{\log s_{k-1}}{\log s_k}}{1 - \frac{\log |I_k|}{\log s_k}}.\end{aligned}$$

The second part of the proposition follows from (3.3) and Theorem 3.2. ☑

As we mentioned above, Lemma 3.1 provides a lower bound for the upper metric mean dimension of a continuous map on the interval. For instance, in [9], Proposition 8, it is proved that if $\varrho : [0,1] \to [0,1]$ satisfies the conditions in Lemma 3.1, with $s_k = k^k$ and $|I_k| = \frac{6}{\pi^2 k^2}$ for any $k \in \mathbb{N}$, then

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varrho) = 1. \tag{6}$$

This fact is a consequence of (4) and (3). Next, it follows from (5) that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varrho) \geq \liminf_{k\to\infty} \frac{\frac{\log s_{k-1}}{\log s_k}}{1 - \frac{\log |I_k|}{\log s_k}} = \liminf_{k\to\infty} \frac{\frac{\log (k-1)^{k-1}}{\log k^k}}{1 - \frac{\log |6/\pi^2 k^2|}{\log k^k}} = 1. \tag{7}$$

Therefore, from (6) and (7) we have

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varrho) = \underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varrho) = 1.$$

We will obtain a continuous map $\varphi_{0,1} : [0,1] \to [0,1]$ such that

$$0 = \underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{0,1}) < \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{0,1}) = 1.$$

Example 3.4. Set $a_0 = 0$ and $a_n = \sum_{i=1}^{n} \frac{6}{\pi^2 i^2}$ for $n \geq 1$. Set $I_n := [a_{n-1}, a_n]$ for any $n \geq 1$. Let $\varphi_{0,1} \in C^0([0,1])$ be defined by

$$\varphi_{0,1}(x) = \begin{cases} T_{n^n}^{-1} \circ g^{n^n} \circ T_{n^n} & \text{if } x \in I_{n^n}, \text{ for } n \geq 1 \\ x & \text{otherwise} \end{cases},$$

where $T_n : I_n := [a_n, a_{n+1}] \to [0,1]$ is the unique increasing affine map from I_n onto $[0,1]$ and $g(x) = |1 - |3x - 1||$ for any $x \in [0,1]$. Each I_{n^n} is an 3^{n^n}-horseshoe for $\varphi_{0,1}$ (see Figure 3). It follows from Theorem 3.2 that

$$\overline{mdim}_M([0,1], |\cdot|, \varphi_{0,1}) = \lim_{n\to\infty} \frac{1}{\left|1 - \frac{\log |I_{n^n}|}{\log 3^{n^n}}\right|} = \lim_{n\to\infty} \frac{1}{\left|1 - \frac{\log\left|\frac{6}{\pi^2 (n^n)^2}\right|}{\log 3^{n^n}}\right|} = 1.$$

Next, we prove that

$$\underline{mdim}_M([0,1], |\cdot|, \varphi_{0,1}) = 0.$$

First observe that since $\varphi_{0,1}$ is the identity outside of $Y = \cup_{j=1}^{\infty} I_{j^j}$, we have

$$\underline{mdim}_M([0,1], |\cdot|, \varphi_{0,1}) = \underline{mdim}_M(Y, |\cdot|, \varphi_{0,1}|_Y).$$

Next, notice that $\frac{\log 3^{(k-1)^{k-1}}}{\log 3^{k^k}} \to 0$ as $k \to \infty$. Hence, for any $\delta > 0$ there exists $k_0 \geq 1$ such that for any $k > k_0$ we have $\frac{\log 3^{(k-1)^{k-1}}}{\log 3^{k^k}} < \delta$. For any $k \geq k_0$, set

$\varepsilon_k = \frac{|I_{k^k}|}{3(3^{k^k})} = \frac{6}{3^{k^k+1}\pi^2 k^{2k}}$. It follows from Corollary 7.2, [7], page 165, for each $j = 1, 2, \ldots, k-1$, we have

$$span(n, \varphi_{0,1}|_{I_{j^j}}, \varepsilon_k) \leq \frac{(3^{j^j})^n}{\varepsilon_k}.$$

Hence, if $Y_k = \cup_{j=1}^{k-1} I_{j^j}$, for every $n \geq 1$ we have

$$span(n, \varphi_{0,1}|_{Y_k}, \varepsilon_k) \leq \sum_{j=1}^{k-1} \frac{(3^{j^j})^n}{\varepsilon_k} \leq \sum_{j=1}^{k-1} \frac{(3^{(k-1)^{k-1}})^n}{\varepsilon_k} \leq (k-1)\frac{(3^{(k-1)^{k-1}})^n}{\varepsilon_k}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{span(\varphi_{0,1}|_{Y_k}, \varepsilon_k)}{n|\log \varepsilon_k|} \leq \limsup_{n\to\infty} \frac{\log\left[(k-1)\frac{(3^{(k-1)^{k-1}})^n}{\varepsilon_k}\right]}{n[\log(3^{k^k+1}\pi^2 k^{2k}/6)]} \leq \frac{\log 3^{(k-1)^{k-1}}}{\log 3^{k^k}}.$$

This fact implies that for any $\delta > 0$ we have

$$\underline{mdim}_M(Y, |\cdot|, \varphi_{0,1}|_Y) < \delta$$

and hence

$$\underline{mdim}_M([0,1], |\cdot|, \varphi_{0,1}) = \underline{mdim}_M(Y, |\cdot|, \varphi_{0,1}|_Y) = 0.$$

Fix $a \in [0,1]$ and let $\phi_a \in C^0([0,1])$ be such that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_a) = \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_a) = a$$

(two different constructions of this kind of maps can be seen in [1, Example 3.1] and [2, Proposition 9.1]). Set $\varphi_{a,1} \in C^0([0,1])$ defined by the formula

$$\varphi_{a,1}(x) = \begin{cases} T_1^{-1} \circ \varphi_{0,1} \circ T_1 & \text{if } x \in [0, \frac{1}{2}], \\ T_2^{-1} \circ \phi_a \circ T_2 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

where $T_1 : [1, \frac{1}{2}] \to [0,1]$ and $T_2 : [\frac{1}{2}, 1] \to [0,1]$ are, respectively, the unique increasing affine map from $[0, \frac{1}{2}]$ onto $[0,1]$ and from $[\frac{1}{2}, 1]$ onto $[0,1]$. We have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{a,1}) = a \quad \text{and} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{a,1}) = 1. \tag{8}$$

Next, for fixed $b \in (0,1)$, we present an example of a continuous map $\varphi_{0,b} \in C^0([0,1])$ such that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{0,b}) = 0 \quad \text{and} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{0,b}) = b.$$

Example 3.5. Fix $r > 0$ and set $b = \frac{1}{r+1}$. Set $a_0 = 0$ and $a_n = \sum_{i=0}^{n-1} \frac{C}{3^{ir}}$ for $n \geq 1$, where $C = \frac{1}{\sum_{i=0}^{\infty} \frac{1}{3^{ir}}} = \frac{3^r - 1}{3^r}$. Let $\varphi_{0,b} \in C^0([0,1])$ be defined by

$$\varphi_{0,b}(x) = \begin{cases} T_{n^n}^{-1} \circ g^{n^n} \circ T_{n^n} & \text{if } x \in I_{n^n}, \text{ for } n \geq 1 \\ x & \text{otherwise} \end{cases},$$

where $T_n : I_n := [a_{n-1}, a_n] \to [0,1]$ is the unique increasing affine map from I_n onto $[0,1]$ and $g(x) = |1 - |3x-1||$ for any $x \in [0,1]$. Each I_{n^n} is a 3^{n^n}-horseshoe for $\varphi_{0,b}$. It follows from Theorem 3.2 that

$$\overline{mdim}_M([0,1], |\cdot|, \varphi_{0,b}) = \lim_{n\to\infty} \frac{1}{\left|1 - \frac{\log |I_{n^n}|}{\log 3^{n^n}}\right|} = \lim_{n\to\infty} \frac{1}{\left|1 + \frac{\log (3^{n^n})^r}{\log 3^{n^n}}\right|} = \frac{1}{1+r} = b.$$

We can prove that

$$\underline{mdim}_M([0,1], |\cdot|, \varphi_{0,b}) = 0$$

just as in Example 3.4.

For $a < b$, let $\phi_a \in C^0([0,1])$ be such that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_a) = \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi_a) = a.$$

Let $\varphi_{a,b} \in C^0([0,1])$ be defined by

$$\varphi_{a,b}(x) = \begin{cases} T_1^{-1} \circ \varphi_{0,b} \circ T_1 & \text{if } x \in [0, \frac{1}{2}], \\ T_2^{-1} \circ \phi_a \circ T_2 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

We have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{a,b}) = a \quad \text{and} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \varphi_{a,b}) = b. \tag{9}$$

We will endow $C^0([0,1])$ with the metric

$$\hat{d}(\phi, \psi) = \max_{x\in[0,1]} |\phi(x) - \psi(x)|.$$

In [9], Proposition 9, it is proved that the set consisting of all $\phi \in C^0([0,1])$ such that $\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = 1$ is dense in $C^0([0,1])$. More generally, in [2], Theorem B, it is proved that, for a fixed $a \in [0,1]$, the set consisting of all continuous maps $\phi \in C^0([0,1])$ such that

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = a$$

is dense in $C^0([0,1])$ (see also [1], Theorem 4.1). Furthermore, we have:

Theorem 3.6. *For $a, b \in [0,1]$, with $a < b$, the set $C_a^b([0,1])$ consisting of all continuous maps $\phi : [0,1] \to [0,1]$ such that*

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = a \quad \textit{and} \quad \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = b$$

is dense in $C^0([0,1])$.

Proof. From (8) and (9) we have for any $a, b \in [0,1]$, with $a \leq b$, there exists $\varphi_{a,b} \in C_a^b([0,1])$. Fix $\psi \in C^0([0,1])$ and take any $\varepsilon > 0$. Given that $C^1([0,1])$ is dense in $C^0([0,1])$, we can assume that $\psi \in C^1([0,1])$ and therefore

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \phi) = 0.$$

Let p^* be a fixed point of ψ. Choose $\delta > 0$ such that $|\psi(x) - \psi(p^*)| < \varepsilon/2$ for any x with $|x - p^*| < \delta$. Take $\varphi_{a,b} \in C_a^b([0,1])$. Set $J_1 = [0, p^*]$, $J_2 = [p^*, p^* + \delta/2]$, $J_3 = [p^* + \delta/2, p^* + \delta]$ and $J_4 = [p^* + \delta, 1]$. Take the continuous map $\psi_{a,b} : [0,1] \to [0,1]$ defined as

$$\psi_{a.b}(x) = \begin{cases} \psi(x), & \text{if } x \in J_1 \cup J_4, \\ T_2^{-1} \varphi_{a,b} T_2(x), & \text{if } x \in J_2, \\ T_3(x), & \text{if } x \in J_3, \end{cases}$$

where $T_2 : J_2 \to [0,1]$ is the affine map such that $T_2(p^*) = 0$ and $T_2(p^* + \delta/2) = 1$, and $T_3 : J_3 \to [p^* + \delta/2, \psi(p^* + \delta)]$ is the affine map such that $T_3(p^* + \delta/2) = p^* + \delta/2$ and $T_3(p^* + \delta) = \psi(p^* + \delta)$. Note that $\hat{d}(\psi_{a,b}, \psi) < \varepsilon$. We have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \psi_{a,b}) = \underline{\mathrm{mdim}}_{\mathrm{M}}(J_2, |\cdot|, \varphi_{a,b}) = a$$

and

$$\overline{\mathrm{mdim}}_{\mathrm{M}}([0,1], |\cdot|, \psi_{a,b}) = \overline{\mathrm{mdim}}_{\mathrm{M}}(J_2, |\cdot|, \varphi_{a,b}) = b,$$

which proves the theorem. ☑

4. Hölder continuous maps with positive metric mean dimension

We say that $\phi : [0,1] \to [0,1]$ is an α-*Hölder continuous map*, for $\alpha \in (0,1]$, if there exists $K > 0$ such that

$$\frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha} \leq K \quad \text{for all } x, y \in [0,1], \text{ with } x \neq y. \tag{10}$$

If ϕ satisfies (10) for $\alpha = 1$, then ϕ is called a *Lipschitz continuous map*.

For $\alpha \in (0,1)$, $H^\alpha([0,1])$ will denote the space of α-Hölder continuous maps on $[0,1]$. $C^1([0,1])$ and $H^1([0,1])$ will denote respectively the space of C^1-maps and the space of Lipschitz continuous maps on $[0,1]$. We have

$$C^1([0,1]) \subset H^1([0,1]) \subset H^\beta([0,1]) \subset H^\alpha([0,1]) \subset C^0([0,1]), \text{where } 0 < \alpha < \beta < 1.$$

Next, suppose that N is a compact Riemannian manifold with topological dimension equal to n and Riemannian metric d. In [1], Theorem 4.5, it is proved that the set consisting of continuous maps on N with *lower* and *upper* metric mean dimension equal to a fixed $a \in [0,n]$ is dense in $C^0(N)$. Furthermore, in Theorem 4.6, it is shown that the set consisting of continuous maps on N with *upper* metric mean dimension equal to n is residual in $C^0(N)$. If $n \geq 2$, in [2], Theorem A, it is proved that the set consisting of homeomorphisms on N with *upper* metric mean dimension equal to n is residual in the set consisting of homeomorphisms on N. It is well known that any homeomorphism on $[0,1]$ has zero entropy and therefore has zero metric mean dimension.

Hence, it remains to show the existence of Hölder continuous maps on finite dimensional compact manifolds. In Theorem 4.5 we will prove that there exist Hölder continuous maps on the interval with positive metric mean dimension, with certain conditions on the Hölder exponent. In Conjectures A, B and C the authors leave three problems that can be objects of future studies.

The next lemma, whose proof is straightforward and left to the reader, will be useful in order to prove that some functions are Hölder continuous.

Lemma 4.1. *Let $\phi : [0,1] \to [0,1]$ be a continuous map such that $\frac{|\phi(x)-\phi(y)|}{|\omega(x-y)|} \leq K$ for some $K > 0$ and any $x \neq y \in [0,1]$, where $\omega(t) = -t\log(t)$ for any $t > 0$ and $\omega(t) = 0$. Then ϕ is an α-Hölder continuous map for any $\alpha \in (0,1)$.*

Definition 4.2. Let $\phi : [0,1] \to [0,1]$ be a continuous map. If there exists $K > 0$ such that $\frac{|\phi(x)-\phi(y)|}{|\omega(x-y)|} \leq K$ for any $x \neq y \in [0,1]$, we will say that ϕ has *modulus of continuity* ω.

Next example, which was introduced in [3], proves that there exist continuous maps with infinite entropy and metric mean dimension equal to zero. That map is also α-Hölder for any $\alpha \in (0,1)$. We will include the details of its construction for the sake of completeness.

Example 4.3. For $n \geq 1$, take $I_n = [2^{-n}, 2^{-n+1}]$. Note that

$$|I_n| = 2^{-n+1} - 2^{-n} = 2^{-n}(2-1) = 2^{-n} \quad \text{for each } n.$$

Divide each interval I_n into $2n+1$ sub-intervals with the same lenght, $I_n^1, \ldots, I_n^{2n+1}$. For $k = 1, 3, \ldots, 2n+1$, let $\phi|_{I_n^k} : I_n^k \to I_n$ be the unique increasing affine map from I_n^k onto I_n and for $k = 2, 4, \ldots, 2n$, let $\phi|_{I_n^k} : I_n^k \to I_n$ be the unique decreasing affine map from I_n^k onto I_n. Note that ϕ is a continuous map ($\phi(x) = x$ for any $x \in \partial I_n$) and each I_n is a $(2n+1)$-horseshoe (see Figure 2), therefore $h_{top}(\phi) = \infty$. It follows from Theorem 3.2 that

$$mdim_M([0,1], |\cdot|, \phi) = 0.$$

We will prove that ϕ is α-Hölder for any $\alpha \in (0,1)$. We will consider the map ω defined in Lemma 4.1.

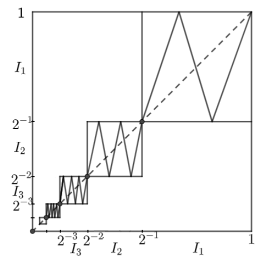


FIGURE 2. Each I_n is a $(2n+1)$-horseshoe

Let $x, y \in [0,1]$ be two distinct points. Then there exist $n \geq 1$ and $m \geq 1$ such that $x \in I_n$ and $y \in I_m$. Given that every I_k is ϕ-invariant, we have that $\phi(x) \in I_n$ and $\phi(y) \in I_m$. We have the next cases:

Case $x = 0$: We have $\phi(0) = 0$ and hence $|\phi(0) - \phi(y)| = |\phi(y)| \leq 2^{-m+1}$. Furthermore,

$$2^{-m} \leq |y| \leq 2^{-m+1} \quad \text{and thus} \quad \frac{1}{|y|} \leq 2^m \quad \text{and} \quad \frac{1}{\log \frac{1}{|y|}} \leq \frac{1}{\log 2^{m-1}}.$$

Hence,

$$\frac{|\phi(y)|}{\omega(|y|)} < \frac{2^{-m+1}2^m}{\log 2^{m-1}} = \frac{2}{(m-1)\log 2}.$$

Case $n > m+1$: In this case we have

$$|\phi(x) - \phi(y)| \leq |2^{-n} - 2^{-m+1}| < 2^{-m+1} \text{ and } 2^{-m+1} > |x-y| > |2^{-n+1} - 2^{-m}| > 2^{-m}.$$

Therefore,

$$\frac{1}{|x-y|} < 2^m \quad \text{and} \quad \frac{1}{\log \frac{1}{|x-y|}} < \frac{1}{\log 2^{m-1}}$$

and thus

$$\frac{|\phi(x) - \phi(y)|}{\omega(|x-y|)} < \frac{2^{-m+1}2^m}{\log 2^{m-1}} = \frac{2}{(m-1)\log 2}.$$

Case $n = m+1$: Note that $I_m = [2^{-m}, 2^{-m+1}]$, $I_{m+1} = [2^{-(m+1)}, 2^{-m}]$ and consider the sub-intervals

$$\begin{aligned} I_m^1 &= \left[2^{-m}, 2^{-m} + \frac{|I_m|}{2m+1}\right] = \left[2^{-m}, 2^{-m} + \frac{2^{-m}}{2m+1}\right] = \left[2^{-m}, \frac{2m+2}{2^m(2m+1)}\right] \\ &= [B_m, C_m] \subseteq I_m \end{aligned}$$

and

$$\begin{aligned} I_{m+1}^{2m+1} &= \left[2^{-m} - \frac{1}{2^{m+1}(2m+3)}, 2^{-m}\right] = \left[\frac{4m+5}{2^{m+1}(2m+3)}, 2^{-m}\right] \\ &= [A_m, B_m] \subseteq I_{m+1}. \end{aligned}$$

For any $x \in I_{m+1}^{2m+1}$, we have

$$\phi(x) = \frac{2^{-m} - 2^{-(m+1)}}{2^{-m} - A_m}(x - 2^{-m}) + 2^{-m} = \frac{2^{-m} - 2^{-(m+1)}}{2^{-m} - \frac{4m+5}{2^{m+1}(2m+3)}}(x - 2^{-m}) + 2^{-m}$$
$$= (2m+3)(x - 2^{-m}) + 2^{-m}.$$

For any $x \in I_m^1$, we have

$$\phi(x) = \frac{2^{-m+1} - 2^{-m}}{C_m - 2^{-m}} \, (x - 2^{-m}) + 2^{-m} = (2m+1)(x - 2^{-m}) + 2^{-m}.$$

Hence, if $x \in I_{m+1}^{2m+1}$ and $y \in I_m^1$, we have

$$\begin{aligned} |x-y| \le C_m - A_m &= \frac{2m+2}{2^m(2m+1)} - \frac{4m+5}{2^{m+1}(2m+3)} \\ &= \frac{1}{2^m}\left[\frac{2m+1}{2m+1} + \frac{1}{2m+1} - \frac{2(2m+3)}{2(2m+3)} + \frac{1}{2(2m+3)}\right] \\ &= \frac{1}{2^m}\left[\frac{1}{2m+1} + \frac{1}{2(2m+3)}\right] \\ &\le \frac{1}{2^m} \; \frac{2}{2m+1} = \frac{1}{2^{m-1}(2m+1)} \end{aligned}$$

and, furthermore,

$$|\phi(y) - \phi(x)| = (2m+1)(y - 2^{-m}) - (2m+3)(x - 2^{-m}) \le (2m+3)(y - x).$$

Therefore,

$$\frac{|\phi(y) - \phi(x)|}{\omega(|x-y|)} \le \frac{(2m+3)(y-x)}{-|x-y| \log |x-y|} \le \frac{2m+3}{\log(2^{m-1}(2m+1))}.$$

Given that

$$\lim_{m\to\infty} \frac{2m+3}{(m-1)\log 2 + \log(2m+1)} = \frac{2}{\log 2},$$

we have

$$\frac{|\phi(y) - \phi(x)|}{\omega(|x-y|)} \le 3. \tag{11}$$

If $x \in I_{m+1}^k$ and $y \in I_m^j$, then there are $x' \in I_{m+1}^{2m+1}$ and $y' \in I_m^1$ such that $\phi(x) = \phi(x')$ and $\phi(y) = \phi(y')$. We have

$$|x'-y'| \le |x-y| \le C_m - A_m = \frac{1}{2^m} \; \frac{2}{2m+1} = \frac{1}{2^{m-1}(2m+1)} < e^{-1}, \text{ for all } m \ge 2.$$

Since $\omega(t)$ is increasing on $[0, e^{-1}]$, it follows that $\omega(|x'-y'|) \le \omega(|x-y|)$ and therefore, by (11), we have

$$\frac{|\phi(y) - \phi(x)|}{\omega(|y-x|)} \le \frac{|\phi(y') - \phi(x')|}{\omega(|y'-x'|)} \le 3.$$

Note also that, if $m=1$, then $\phi|_{I_1\cup I_2}$ is a Lipschitz function. Hence, in particular, it has modulus of continuity ω.

Case $n=m$: In this case we have that there exists y' in the same branch as x such that $f(y')=f(y)$ and $|x-y|\geq |x-y'|$. Note that

$$|x-y'|\leq \frac{|I_n|}{2n+1}=\frac{1}{2^n(2n+1)} \text{ and hence } \frac{1}{\log(|x-y'|^{-1})}\leq \frac{1}{\log(2^n(2n+1))}.$$

Therefore,

$$\frac{|\phi(x)-\phi(y)|}{\omega(|x-y|)}\leq \frac{|\phi(x)-\phi(y')|}{\omega(|x-y'|)}=\frac{(2n+1)|x-y'|}{|x-y'|\log(|x-y'|^{-1})}=\frac{2n+1}{\log(2^n(2n+1))}.$$

In each case we have that $\frac{|\phi(x)-\phi(y)|}{\omega(|x-y|)}$ is bounded. Therefore, ϕ is an α-Hölder continuous map for any $\alpha\in(0,1)$.

We recall the map ϱ presented in (6) has (lower and upper) metric mean dimension equal to 1. In that case, for any $k\geq 1$, ϱ has a k^k-horseshoe with length equal to $\frac{6}{\pi^2k^2}$. We can prove that ϱ is not α-Hölder for none $\alpha\in(0,1)$, because the number of legs of each horseshoe is so big, and therefore the slopes of the map on each subinterval increases very quickly. Next, we will present a path (depending of the length of the horseshoes) consisting of continuous maps such that each of them have less legs than ϱ, their metric mean dimension is equal to one, however, they are not α-Hölder for none $\alpha\in(0,1)$.

Example 4.4. Fix any $\beta\in(1,\infty)$ and take $g\in C^0([0,1])$, defined by $x\mapsto |1-|3x-1||$. Set $a_n=\sum_{k=1}^n\frac{C}{k^\beta}$ for $n\geq 1$, where $C=\frac{1}{\sum_{k=1}^\infty k^{-\beta}}$. For each $n\geq 2$, let $T_n: I_n:=[a_{n-1},a_n]\to[0,1]$ be the unique increasing affine map from I_n onto $[0,1]$. Consider the map $\phi_\beta:[0,1]\to[0,1]$ defined by

$$\phi_\beta|_{I_n}=T_n^{-1}\circ g^n\circ T_n \quad \text{ for each } n\geq 2.$$

Hence, each I_n, whose length is $\frac{C}{n^\beta}$, is a 3^n-horseshoe for ϕ. From Theorem 3.2 and Proposition 3.3 we have

$$\underline{mdim}_M([0,1],|\cdot|,\phi_\beta)=\overline{mdim}_M([0,1],|\cdot|,\phi_\beta)=1.$$

Hence, $\{\phi_\beta\}_{\beta\in(1,\infty)}$ is a path of continuous map with metric mean dimension equal to 1. Next, we will prove that ϕ is not α-Hölder for none $\alpha\in(0,1)$. Note that each I_n can be divided into 3^n sub-intervals with the same length, $I_{n,1},I_{n,2},\dots,I_{n,3^n}$ such that for any $j\in\{1,2\dots,3^n\}$ we have

$$\phi_\beta(x)=\begin{cases}3^n(x-a_{n-1})-(j-1)(a_n-a_{n-1})+a_{n-1} & \text{if } x\in I_{n,j} \text{ for } j \text{ odd}\\ -3^n(x-a_{n-1})+(j-1)(a_n-a_{n-1})+a_n & \text{if } x\in I_{n,j} \text{ for } j \text{ even.}\end{cases}$$

Hence, for each $x, y \in I_{n,j}$, we obtain that

$$|x-y| \leq |I_{n,j}| = \frac{|I_n|}{3^n} = \frac{C}{3^n[n^\beta]} \quad \text{and} \quad |\phi_\beta(x) - \phi_\beta(y)| = 3^n|x-y|.$$

Therefore, if $x = a_{n-1} \in I_{n,1}$ and $y = a_{n-1} + \frac{a_n - a_{n-1}}{3^n} \in I_{n,1}$, we have

$$\frac{|\phi_\beta(x) - \phi_\beta(y)|}{|x-y|^\alpha} = 3^n|x-y|^{1-\alpha} \leq \frac{3^n C^{1-\alpha}}{3^{(1-\alpha)n}[n^\beta]^{(1-\alpha)}} = \frac{3^{\alpha n} C^{1-\alpha}}{[n^\beta]^{(1-\alpha)}} \to \infty$$

as $n \to \infty$. Thus, ϕ_β is not α-Hölder.

In the example above we can see that, for every β and γ in $(1, \infty)$, we have that ϕ_β and ϕ_γ are topologically conjugate and they have the same metric mean dimension. As we mentioned in Remark 2.5, the metric mean dimension is not invariant under topological conjugacy. In [1], Remark 3.2, the authors construct a path of continuous maps, $\{\phi_\beta\}_{\beta \in (0,\infty)}$, such that for every β and γ in $(0, \infty)$, ϕ_β and ϕ_γ are topologically conjugate, however,

$$\text{mdim}_{\text{M}}([0,1], |\cdot|, \phi_\beta) \neq \text{mdim}_{\text{M}}([0,1], |\cdot|, \phi_\gamma) \quad \text{if } \beta \neq \gamma.$$

In the next theorem we prove the existence of Hölder continuous maps on the interval which have positive metric mean dimension. Note that for the map presented in Example 4.3, the increase of the number of legs, which is $2n+1$ for each $n \geq 1$, is very slow compared to the decrease in size of the horseshoes, which is $\frac{1}{2^n}$ for each $n \geq 1$. Therefore, it follows from Theorem 3.2 and Proposition 3.3 that map has metric mean dimension equal to zero. If we keep the same subintervals $I_n = [2^{-n}, 2^{-n+1}]$ and we wish to obtain a continuous map on $[0, 1]$ with positive metric mean dimension, we must have $\limsup_{n\to\infty} \frac{\log 2^n}{\log s_n} < \infty$, where s_n is the number of legs of the map on each I_n. This fact implies that s_n must increase very quickly and therefore, the Hölder exponent of the map must be zero or very small. Hence, we must modify the size of the horseshoe in order to obtain a continuous map with positive metric mean dimension and a greater Hölder exponent: the number of legs must not increase so fast (otherwise the Hölder exponent decreases) and the size of the horseshoe must not decay too fast (otherwise the metric mean dimension decreases). In the next theorem, we present an example where the size of the horseshoe is $\frac{C}{3^{nr}}$, for a fixed $r \in (0, \infty)$ and a constant $C > 0$, and the number of legs is 3^n, for any $n \in \mathbb{N}$. These sequences, $\frac{C}{3^{nr}}$ and 3^n, are reasonably similar and we will prove that in this case the map has positive metric mean dimension and a considerable Hölder exponent.

Theorem 4.5. *Fix $a \in [0, 1)$ and take $\alpha = 1 - a$. Then, there exists an α-Hölder continuous map $\phi_a : [0, 1] \to [0, 1]$ such that*

$$\underline{mdim}_M([0,1], |\cdot|, \phi_a) = \overline{mdim}_M([0,1], |\cdot|, \phi_a) = a.$$

Proof. If $a = 0$ we can take ϕ_0 as the identity on $[0,1]$. Fix $r > 0$ and let $a = \frac{1}{r+1}$. Set $a_0 = 0$ and $a_n = \sum_{i=0}^{n-1} \frac{C}{3^{ir}}$ for $n \geq 1$, where $C = \frac{1}{\sum_{i=0}^{\infty} \frac{1}{3^{ir}}} = \frac{3^r - 1}{3^r}$. For each $n \geq 1$, set $I_n = [a_{n-1}, a_n]$ and take $T_n : I_n \to [0,1]$ and g as in Example 4.4. Consider the function $\phi_a : [0,1] \to [0,1]$, given by $\phi_\alpha|_{I_n} = T_n^{-1} \circ g^n \circ T_n$ for any $n \geq 1$. It follows from Theorem 3.2 that

$$\underline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi_a) = \overline{\text{mdim}}_{\text{M}}([0,1], |\cdot|, \phi_a) = a.$$

We will prove that $\phi_a : [0,1] \to [0,1]$ is α-Hölder with $\alpha = \frac{r}{r+1} = 1 - a$. Let $x, y \in [0,1]$ be two distinct points. Thus, there exist $n \geq 1$ and $m \geq 1$ such that $x \in I_n$ and $y \in I_m$. Given that each I_k is ϕ_a-invariant, we have $\phi_a(x) \in I_n$ and $\phi_a(y) \in I_m$. We have the following cases:

Case $x = 0$: Since $\phi_a(0) = 0$, we have

$$|\phi_a(0) - \phi_a(y)| = |\phi_a(y)| \leq \frac{1 - 3^{-mr}}{1 - 3^{-r}}.$$

Furthermore, if $y \in I_m$, we obtain that

$$\frac{|\phi(y)|}{\omega(|y|)} < \frac{1 - 3^{-mr}}{[1 - 3^{(-m+1)r}] \log \frac{1 - 3^{-mr}}{1 - 3^{-r}}},$$

where ω is the map defined in Lemma 4.1. The right side of the last inequatily converges to $\frac{1}{\log \frac{1}{1-3^{-r}}}$. Thus $\frac{|\phi(y)|}{\omega(|y|)}$ is bounded.

Case $n = m + k$**, with** $k > 1$ **for a fixed** m: we know that

$$|\phi_a(x) - \phi_a(y)| \leq \frac{3^{(1-m)r} - 3^{-nr}}{1 - 3^{-r}} = \frac{3^{(n-m+1)r} - 1}{3^{nr}(1 - 3^{-r})} \text{ and } \frac{1}{|x - y|} \leq \frac{(1 - 3^{-r})3^{nr}}{3^{(n-m)r} - 1},$$

hence

$$\begin{aligned} \frac{|\phi_a(x) - \phi_a(y)|}{|x - y|^\alpha} &\leq \frac{3^{nr\alpha}(1 - 3^{-r})^{\alpha-1}[3^{(n-m+1)r} - 1]}{3^{nr}[3^{(n-m)r} - 1]^\alpha} \\ &\leq 3^{nr(\alpha-1)}(1 - 3^{-r})^{\alpha-1} \frac{(3^{kr+r-kr\alpha} - 3^{-kr\alpha})}{\left(1 - \frac{1}{3^{kr}}\right)^\alpha}. \end{aligned} \tag{12}$$

Since $\lim_{k\to\infty} \left(1 - \frac{1}{3^{kr}}\right) = 1$, we have that the right side of (12) converges to $(1 - 3^{-r})^{\alpha-1} 3^{m(\alpha-1)r+r}$, as $k \to \infty$. Thus, $\frac{|\phi_a(x) - \phi_a(y)|}{|x-y|^\alpha}$ is bounded with $x \in I_n$, $y \in I_m$, $n > m + 1$.

Case $n = m + 1$: note that $I_m = [a_{m-1}, a_m]$, $I_{m+1} = [a_m, a_{m+1}]$,

$$I_m = I_m^1 \cup \cdots \cup I_m^{3^m}; \quad I_{m+1} = I_{m+1}^1 \cup \cdots \cup I_{m+1}^{3^{m+1}} \quad \text{and} \quad I_m \cap I_{m+1} = \{a_m\},$$

where $|I_m^j| = \frac{|I_m|}{3^m}$ and $|I_{m+1}^j| = \frac{|I_{m+1}|}{3^{m+1}}$. Suppose that $x \in I_{m+1}^1$ and $y \in I_m^{3^m}$. Hence

$$|x-y| \leq a_m + \frac{|I_{m+1}|}{3^{m+1}} - \left(a_m - \frac{|I_m|}{3^m}\right) = \frac{|I_{m+1}|}{3^{m+1}} + \frac{|I_m|}{3^m} = \frac{C(1+3^{r+1})}{3^{m(r+1)+1}}. \quad (13)$$

It is easy to check that

$$\phi_a(y) = 3^m(y-a_m) + a_m \quad \text{and} \quad \phi_a(x) = 3^{m+1}(x-a_m) + a_m.$$

Hence, as $y \leq a_m$, we have that

$$\begin{aligned}|\phi_a(x)-\phi_a(y)| &= |3^{m+1}(x-a_m) - 3^m(y-a_m)| = 3^{m+1}x - 3^m y - (3^{m+1}-3^m)a_m\\ &\leq 3^{m+1}x - 3^m y - (3^{m+1}-3^m)y = 3^{m+1}(x-y).\end{aligned}$$

Therefore, from (13) we obtain that

$$\frac{|\phi_a(x)-\phi_a(y)|}{|x-y|^\alpha} \leq 3^{m+1}|x-y|^{1-\alpha} \leq 3^{m(\alpha(r+1)-r)}3^\alpha(C(1+3^{r+1}))^{1-\alpha}. \quad (14)$$

To finish this case, we assume that $x \in I_{m+1}^j$ and $y \in I_m^i$ for some j and i. Then, there are $x' \in I_{m+1}^1$ and $y' \in I_m^{3^m}$ such that $\phi_a(x) = \phi_a(x')$ and $\phi_a(y) = \phi_a(y')$. Moreover, $|x-y| \geq |x'-y'|$, which provides

$$\begin{aligned}|\phi_a(x)-\phi_a(y)| &= |\phi_a(x')-\phi_a(y')| \leq 3^{m(\alpha(r+1)-r)}3^\alpha(C(1+3^{r+1}))^{1-\alpha}|x'-y'|^\alpha\\ &\leq 3^{m(\alpha(r+1)-r)}3^\alpha(C(1+3^{r+1}))^{1-\alpha}|x-y|^\alpha. \quad (15)\end{aligned}$$

Since $\alpha = \frac{r}{1+r}$, then both equations (14) and (15) are bounded as $m \to \infty$.

Case $n = m$: since $\phi_a(x) = 3^n(x-a_n) + a_n$ for any $x \in I_n^{3n}$, we have that $|\phi_a(x)-\phi_a(y)| = 3^n|x-y|$, for all $x, y \in I_n^{3n}$. Therefore,

$$\begin{aligned}\frac{|\phi_a(x)-\phi_a(y)|}{|x-y|^\alpha} &= 3^n|x-y|^{1-\alpha} \leq 3^n\left(\frac{|I_n|}{3^n}\right)^{1-\alpha} = 3^n\left(\frac{C}{3^{(n-1)r}3^n}\right)^{1-\alpha}\\ &= C^{1-\alpha}3^{r-\alpha r}3^{n(\alpha-r+\alpha r)},\end{aligned}$$

which is bounded, since $\alpha \leq \frac{r}{1+r}$.

In conclusion, we prove that ϕ_a is $(1-a)$-Hölder. ☑

Now we state an immediate consequence of the last theorem. Let X be a compact metric space with metric d. If $\phi_1, \dots, \phi_n \in C^0(X)$ and

$$\overline{\text{mdim}}_{\text{M}}(X, d, \phi_k) = \underline{\text{mdim}}_{\text{M}}(X, d, \phi_k) = a_k, \quad \text{for any } k = 1, \dots, n,$$

we have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(X \times \cdots \times X, d^n, \phi_1 \times \cdots \times \phi_n) =$$
$$\overline{\mathrm{mdim}}_{\mathrm{M}}(X \times \cdots \times X, d^n, \phi_1 \times \cdots \times \phi_n) = \sum_{k=1}^{n} a_k,$$

where

$$d^n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = d(x_1, y_1) + \cdots + d(x_n, y_n), \quad \text{for } x_1, \ldots, x_n, y_1, \ldots, y_n \in X.$$

(see [1], Theorem 3.13, v). Hence, it follows from Theorem 4.5 that:

Corollary 4.6. *Fix $b \in [0, n]$ and $\alpha = 1 - \frac{b}{n}$. Then, there exists $\psi_b \in H^{\alpha}([0,1]^n)$ such that*

$$\overline{mdim}_M([0,1]^n, d^n, \psi_b) = \underline{mdim}_M([0,1]^n, d^n, \psi_b) = b.$$

***Proof*.** Take $\phi_a : [0,1] \to [0,1]$ defined in the proof of Theorem 4.5, where $a = \frac{b}{n}$, and set $\psi_b = \phi_a \times \cdots \times \phi_a : [0,1]^n \to [0,1]^n$. We have

$$\underline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, d^n, \psi_b) = \overline{\mathrm{mdim}}_{\mathrm{M}}([0,1]^n, d^n, \psi_b) = na = b.$$

Given that each ϕ_a is an $(1 - \frac{b}{n})$-Hölder continuous map, we have that ψ_b is an $(1 - \frac{b}{n})$-Hölder continuous map. ☑

5. Some conjectures about this research

Note that in Theorem 4.5 there exists a relationship between the Hölder exponent α and the metric mean dimension of a continuous map on the interval. We will present the next conjectures about this relationship which can be the subject of future research.

Conjecture A. If $\phi : [0,1] \to [0,1]$ is an α-Hölder continuous map, then

$$mdim_M([0,1], |\cdot|, \phi) \leq 1 - \alpha.$$

Note that in Theorem 4.5 we prove that there exists an α-Hölder continuous map $\phi : [0,1] \to [0,1]$ with $\mathrm{mdim}_{\mathrm{M}}([0,1], |\cdot|, \phi) = 1 - \alpha$, where $\alpha \in (0,1)$. From (2), we have that there is no continuous map on the interval with metric mean dimension bigger than 1.

Conjecture B. There is no α-Hölder continuous map $\phi : [0,1] \to [0,1]$, with $\alpha > 0$ satisfying

$$mdim_M([0,1], |\cdot|, \phi) = 1.$$

Note that if Conjecture A is true, then Conjecture B is a consequence of Conjecture A.

Conjecture C. If $\phi : [0,1] \to [0,1]$ is an α-Hölder continuous map for any $\alpha \in (0,1)$, then

$$mdim_M([0,1], |\cdot|, \phi) = 0.$$

Note if Conjecture A is true, then Conjecture C is a consequence of Conjecture A.

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Facultad de Ciencias Básicas
Universidad Tecnológica de Bolívar,
Cartagena de Indias - Colombia
e-mail: jmuentes@utb.edu.co

Universidade Federal do Rio de Janeiro,
Rio de Janeiro - Brasil
e-mail: sergiori@im.ufrj.br

Universidade Federal de Maranhão,
Campus Balsas, Brasil
e-mail: raibel.jac@ufma.br