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Nonlinear analysis and control of a reaction wheel pendulum: Lyapunov-based approach

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ABSTRACT

This paper presents a nonlinear analysis, control, and comparison of controllers based on the dynamical model of the reaction wheel pendulum (RWP) in a tutorial style. Classical methodologies such as proportional integral derivative (PID) control and state variables feedback control are explored. Lyapunov's method is proposed to analyze the stability of the proposed nonlinear controllers, and it is also used to design control laws guaranteeing globally asymptotically stability conditions in closed-loop. A swing up strategy is also included to bring the pendulum bar to the desired operating zone at the vertical upper position from an arbitrary initial location. Simulation results show that it is possible to obtain the same dynamical behavior of the RWP system adjusting the control gains adequately. All simulations were conducted via MATLAB Ordinary Differential Equation packages.

1. Introduction

Test prototypes are used to validate diverse strategies of linear and nonlinear controls [7]. This is due to the fact that their nonlinear dynamics allows understanding the phenomena and physical behavior of factories and equipment in practical applications, such as robotic systems (e.g., snake-type robots, flexible-link robots, mobile robots, and walking robots), aerospace systems (e.g., helicopters, aircraft, spacecraft, satellites, and rockets), or marine vehicles (e.g., surface vessels and underwater vehicles) [2,9], among others. There are different variants of the classic models of the pendulum, such as the reaction wheel pendulum, the pendulum on a cart with linear displacement, pendulum models with two and three bars, the Furuta pendulum with a rotating base, Pendubot, and Acrobot, among others [12,11,5].

The reaction wheel pendulum (RWP) was introduced by Spong [19]. It is a variant of the inverted pendulum, which has a bar which can spin freely around the support point (pivot) at one of its ends, as is depicted in Fig. 1. The RWP has a motor coupled to the opposite end of the pivot, acting on a wheel of inertia with which the oscillations of the wheel are controlled, due to the reaction torque τ. The angle φ of the pendulum (from the vertical) and the angle α between the pendulum and the wheel are measured with sensors located on each of the axes of rotation.

The RWP presents basically two problems: the first is maintaining the local stability around the equilibrium position, which is analogous to the problem of the juggler who intends to keep a stick on the tip of a finger. Several control strategies have been proposed for this problem. In [15], a proportional integral controller is introduced. A fuzzy-logic approach is presented in [3], a passivity-based control is described in [17], and feedback linearization is employed in [19].

The second problem is that of lifting the pendulum from its rest position to the upright position. This problem is known as "swing up" [19]. To address this problem, the most used strategies are based on a gradual increase with oscillations of increasing amplitude, in which there stand out trajectory tracking [3] and energy regulation [14,13].

While it is true that this system has been widely studied using linear and nonlinear techniques, it is important to mention that the use of linear techniques requires the application of linearization methods such as Taylor's series or trigonometric approximations [15]. These techniques can operate well around the operating point; nevertheless, they only guarantee local stability and their performance decreases speedily when the pendulum moves away from the operating point [3]. In the case of nonlinear methodologies, some of them are based on artificial intelligent like neural networks or fuzzy logic developments [21]. Although their implementation does not require a dynamical model of the system,

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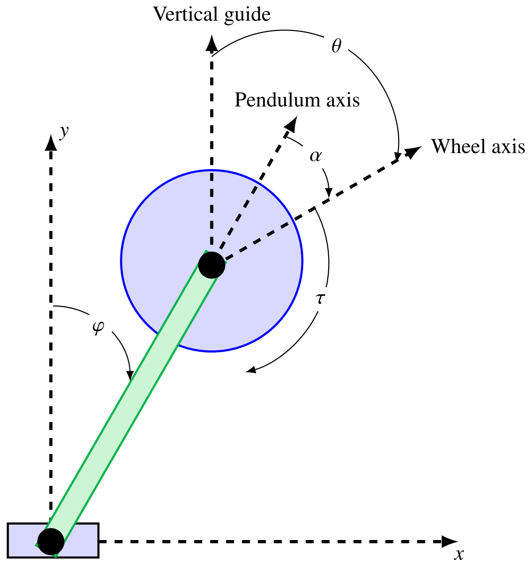

Fig. 1. Bi-dimensional RWP scheme.

it is hard to guarantee the stability properties in closed-loop. Additionally, these techniques can cause undesired oscillations around the operating point [14]. On the other hand, there are nonlinear strategies, such as exact or partial feedback linearizations or passivity-based approaches, that work well enough over the entire range of operation and also guarantee global stability conditions in the sense of Lyapunov [2].

Unlike the aforementioned papers, the present paper gives the design of a controller to operate an RWP system by applying Lyapunov theory in a direct way. The main advantage of this is the possibility of guaranteeing stable operation through the control input without requiring extra calculations. In addition, at least two different Lyapunov functions are presented in the controller design in conjunction with a swing up strategy to bring the position of the bar pendulum, starting from any location, closer to the desired operating point [1].

The proposed controller strategy is compared with classical linear and nonlinear techniques, showing its efficiency and robustness. All simulation results were obtained using MATLAB, using the ordinary differential equation packages ODE/45-15s-23tb.

Note that one of the main contributions of this paper corresponds to its tutorial style, since it provides multiple forms of developing control strategies for nonlinear systems, which allows engineering students to understand the application of both linear control approaches (PID and feedback realizations) and nonlinear methods (exact feedback realization and Lyapunov's direct methods) on complex nonlinear systems.

The remainder of this paper is organized as follows. Section 2 shows the development of the dynamical model for the RWP as well as its reduced equivalent model. Section 3 explores the basic ideas of the Lyapunov theory of stability analysis of equilibrium points. Section 4 shows the six studied controllers for addressing the problem of regulating the position of the RWP, by presenting conventional PID and feedback control realizations for the equivalent Taylor's model around the operating point. In addition, there are included those realizations based on exact feedback linearization method as well as two control Lyapunov functions. Section 5 presents all the simulation results, including model uncertainties. The main conclusions derived from this research are presented in the last section.

2. Dynamical model of an RWP

Typically, Lagrangian or Newtonian mechanics is employed in the specialized literature to obtain the dynamical model of a pendulum system [2]. The basic diagram of the physical system under study is depicted in Fig. 1.

Defining $\theta = \varphi + \alpha$, the dynamical model of the RWP system can be written as follows:

$$\begin{aligned} \ddot{\varphi} &= a\sin(\varphi) - bu, \\ \ddot{\theta} &= cu, \end{aligned} \tag{1}$$

where a, b and c are constants related to the physical parameters of the system, φ represents the angular position of the pendulum measured from the vertical axis, and θ is the relative angle of the reaction wheel measured from the same vertical reference.

To transform the set of Eq. (1) to a state-space representation, the following state variables will be used: $x_1 = \varphi, x_2 = \dot{x}_1$ and $x_3 = \dot{\theta}$. After substituting these into (1), one obtains

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a\sin(x_1) - bu, \\ \dot{x}_3 &= cu. \end{aligned} \tag{2}$$

It should be noted that the dynamics of the angular speed of the reaction wheel depends exclusively on the control input. For this reason its dynamics is completely determined by the control behavior, i.e., if the input signal u is stable bounded, the third term of (2) can be solved for:

$$x_3 = c\int_{t_0}^{t} u(s)ds. \tag{3}$$

Taking into account the above mentioned reduction, the simplified dynamical model of the RWP system can be expressed as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a\sin x_1 - bu. \end{aligned} \tag{4}$$

2.1. Equilibrium points

Note that the equilibrium points of the RWP dynamical model defined by (4) can be easily obtained by solving these equations [16]:

$$\begin{aligned} x_2 &= 0, \\ \sin(x_1) &= a^{-1}bu. \end{aligned} \tag{5}$$

The second equality of (5) shows that the equilibrium points of the angular position for the pendulum depend exclusively on the control input; nevertheless, if one analyzes the third equation of (2), the admissible values of the external input to guarantee that the reaction wheel reaches its equilibrium point occur only when the external input is strictly equal to zero. To satisfy this condition, it is necessary that x_1 be equal to $n\pi$, where $n \in \mathbb{Z}$.

2.2. Stability analysis

The reduced dynamical system defined by (4) can be represented through its open-loop dynamics. That is, $u = 0$, when the main interest is to characterize the nature of its equilibrium points, which produces the following structure [8]:

$$\begin{aligned} &\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \mathcal{V}_{x_1}(x_1, x_2) \\ \mathcal{V}_{x_2}(x_1, x_2) \end{pmatrix} \\ &\mathcal{V}_{x_i} = \tfrac{\partial}{\partial x_i}\mathcal{V}(x_1, x_2) \\ &\mathcal{V}(x_1, x_2) = 2a\cos^2\left(\tfrac{1}{2}x_1\right) + \tfrac{1}{2}x_2^2. \end{aligned} \tag{6}$$

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Note that the function $\mathcal{V}(x_1,x_2)$ corresponds to a possible energy storage function of the system. Analyzing this energy storage function, it is possible to see that if $\mathcal{V}(x_1,x_2)$ has a local minimum in (x_1^*,x_2^*), then (x_1^*,x_2^*) is stable. Otherwise, if $\mathcal{V}(x_1,x_2)$ has a saddle point or a local maximum in (x_1^*,x_2^*), then these points are unstable.

Applying the gradient operator to the energy storage function yields

$$\begin{pmatrix} \mathcal{V}_{x_1}(x_1,x_2) \\ \mathcal{V}_{x_2}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} -a\sin(x_1) \\ x_2 \end{pmatrix} \tag{7}$$

Note that the critical points of the energy storage function correspond to the equilibrium points of the reduced RWP model: $x_1 = n\pi$ and $x_2 = 0$.

Taking the Jacobian of the system of Eqs. (7), one obtains the following matrix:

$$\mathcal{J}(x_1,x_2) = \begin{bmatrix} \mathcal{V}_{x_1x_1} & \mathcal{V}_{x_1x_2} \\ \mathcal{V}_{x_2x_1} & \mathcal{V}_{x_2x_2} \end{bmatrix} = \begin{bmatrix} -a\cos(x_1) & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

Definition 1. If the function $\mathcal{V}(x_1,x_2)$ has a critical point in (x_1^*,x_2^*), and $\mathcal{J}(x_1,x_2)$ is its Jacobian matrix, then (x_1^*,x_2^*) can be classified by using the eigenvalues of $\mathcal{J}(x_1,x_2)$ as follows:

1. If $\mathcal{J}(x_1,x_2)$ is positive definite, that is, $\lambda_1 > 0$ and $\lambda_2 > 0$ with $\lambda_1,\lambda_2 \in \mathbb{R}$, then (x_1^*,x_1^*) is a local minimum.
2. If $\mathcal{J}(x_1,x_2)$ is positive definite, that is, $\lambda_1 < 0$ and $\lambda_2 < 0$ with $\lambda_1,\lambda_2 \in \mathbb{R}$, then (x_1^*,x_1^*) is a local maximum.
3. If $\mathcal{J}(x_1,x_2)$ is neither positive nor negative definite, that is, if either $\lambda_1 > 0$ and $\lambda_2 < 0$ or $\lambda_1 < 0$ and $\lambda_2 > 0$ with $\lambda_1,\lambda_2 \in \mathbb{R}$, then (x_1^*,x_1^*) is a saddle point.
4. If $\mathcal{J}(x_1,x_2)$ has complex eigenvalues, then this criterion is indecisive.

Note that the eigenvalues of $\mathcal{J}(x_1,x_2)$ can be calculated as follows:

$$\lambda_1 = -a\cos(x_1) \quad \wedge \quad \lambda_2 = 1. \tag{9}$$

When (9) is observed, then on applying the Definition 1, it is easy to demonstrate that

$$\begin{aligned} &\left.\begin{aligned} x_1 &= 2k\pi \\ x_2 &= 0 \end{aligned}\right\} k \in \mathbb{Z} \quad \text{is a saddle point} \rightarrow \text{unstable} \\ &\left.\begin{aligned} x_1 &= (2k+1)\pi \\ x_2 &= 0 \end{aligned}\right\} k \in \mathbb{Z} \quad \text{is a local minimum} \rightarrow \text{stable} \end{aligned} \tag{10}$$

Based on the results presented in (10), we are mainly interested in obtaining a general control law u such that the saddle point $(x_1^*,x_2^*) = ((2k+1)\pi, 0)$ with $k \in \mathbb{Z}$ becomes a stable point.

3. Basic ideas of Lyapunov-based control theory

Lyapunov stability theory is a standard tool and one of the most important tools in the analysis of nonlinear systems [8]. We consider the following nonlinear autonomous system:

$$\dot{x} = f(x), \tag{11}$$

where $f : \mathcal{O} \longrightarrow \mathcal{R}^n$ is a locally Lipschitz map from the domain $\mathcal{O} \subseteq \mathcal{R}^n$to $\mathcal{R}^n$. Suppose that the system shown in (11) has an equilibrium point in $\bar{x} \in \mathcal{O}$ (i.e., $f(\bar{x}) = 0$). The question then arises, how to know if the equilibrium point $\bar{x}$ is stable. First, we assume that $\bar{x}$ is the origin of the state space. This does not involve any loss of generality since we can always apply a change of variables $\dot{\xi} = x - \bar{x}$ to obtain [22]:

$$\dot{\xi} = f(\xi + \bar{x}) \equiv g(\xi). \tag{12}$$

Now, the stability is studied for the new system with respect to $\xi = 0$. There are two types of stability [8].

Definition 2. The equilibrium point $x = 0$ of (11) is

1. *Stable* if, for each $\epsilon > 0$, there exists a $\alpha = \alpha(\epsilon) > 0$such that
$$\|x(t_0)\| < \alpha \Rightarrow \|x(t_0)\| < \epsilon, \quad \forall\, t > t_0. \tag{13}$$
2. *Asymptotically stable* if it is stable and α can be chosen such that
$$\|x(t_0)\| < \alpha \Rightarrow \lim_{t\longrightarrow\infty} x(t) = 0. \tag{14}$$

Also, it defined to be unstable if it is not stable.

3.1. Lyapunov's direct method

This method tries to determine the stability directly by means of functions which are defined in the state space [8].

Theorem 1. [Lyapunov's Theorem] *Considers the system (11) and suppose that there is a continuously differentiable function* $V : \mathcal{O} \longrightarrow \mathcal{R}$ *such that*

$$\begin{aligned} V(0) &= 0, \\ V(x) &> 0, \qquad x \in \mathcal{O}\ \forall\, x \neq 0 \\ \dot{V}(x) &= \tfrac{\partial V}{\partial x} f(x) \leqslant 0, \quad x \in \mathcal{O}. \end{aligned} \tag{15}$$

If the above is true, the equilibrium point is stable in the sense of Lyapunov. If also

$$\dot{V}(x) = \tfrac{\partial V}{\partial x} f(x) < 0, \quad x \in \mathcal{O}, \tag{16}$$

the equilibrium point is asymptotically stable. This stability will be global if $\mathcal{O} = \mathcal{R}^n$ *and, in addition, the function V is radially unbounded, i.e.,* $\lim_{\|x\|\longrightarrow\infty} V(x) = \infty$. *Finally, if there exist scalars* $\alpha, \beta, \gamma > 0$ *and* $p \geqslant 1$ *such that*

$$\begin{aligned} &\alpha\|x\|^p \leqslant V(x) \leqslant \beta\|x\|^p, \qquad x \in \mathcal{O} \\ &\dot{V}(x) = \tfrac{\partial V}{\partial x} f(x) \leqslant \gamma V(x), \qquad x \in \mathcal{O} \end{aligned} \tag{17}$$

the equilibrium point is exponentially stable. This stability will be global if $\mathcal{O} = \mathcal{R}^n$ *and, in addition, the function V is radially unbounded [8].*

Fig. 2 illustrates the differences between stability in the sense of Lyapunov, asymptotic stability, and instability.

4. Design of a general control law for the RWP

There are several approaches to designing controllers for an RWP using a dynamic model of the system. We have selected some classical linear and nonlinear methodologies to compare with our proposed Lyapunov methodology. This section presents the classical PID design, feedback control via linearization, exact feedback linearization using pole reassignment, and the Lyapunov control strategy.

4.1. Linear controllers

4.1.1. Classical PID approach

While it is true that PID control can be applied to linear and nonlinear systems with good results, this kind of control is usually employed for linear systems, mainly when the system has one particular variable of interest and one control input [5,3]. In case of the

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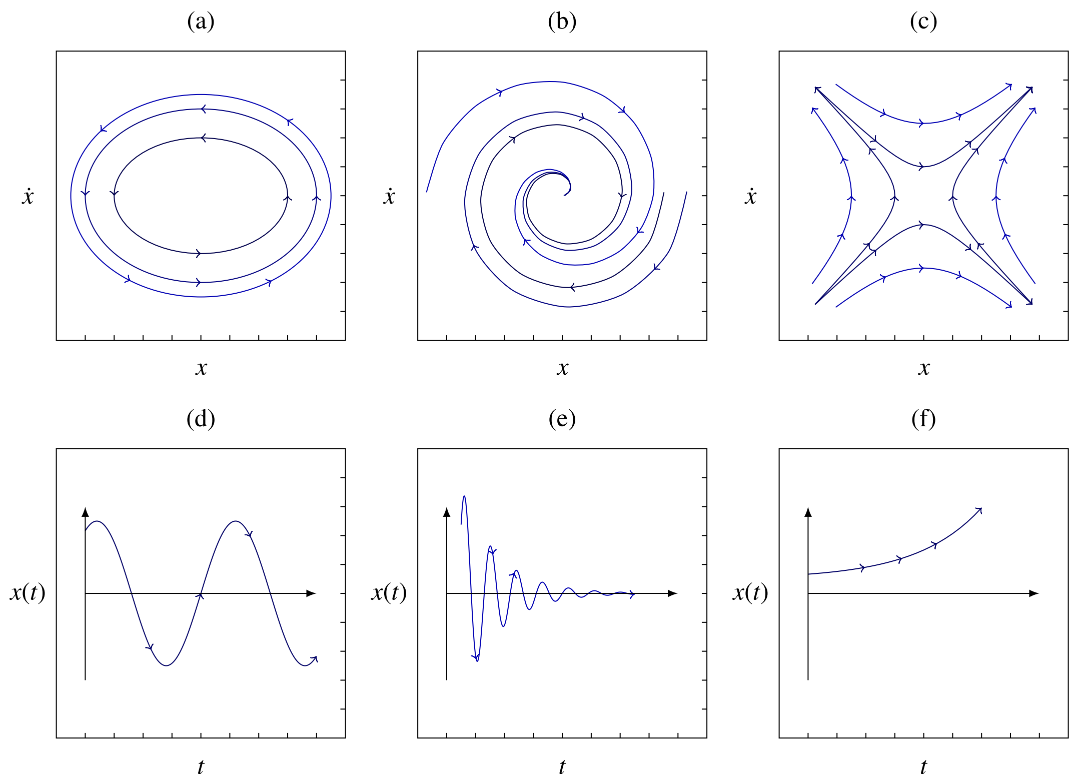

Fig. 2. Phase portrait for stable and unstable equilibrium points: (a) Stable in the sense of Lyapunov, (b) Asymptotically stable, (c) Unstable (saddle), (d) time domain for a stable equilibrium point, (e) time domain for an asymptotically stable equilibrium point, and (f) dynamical behavior of an unstable equilibrium point.

reduced RWP model, the interest is to control the position of the pendulum x_1 using the control input u.

From the reduced dynamic system presented by (4) there can be obtained a linear representation around the upright position by applying a linearization using Taylor's series as follows:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \end{bmatrix} \Delta u \tag{18}$$

The classical structure of the PID controller is

$$\Delta u = k_p e(t) + k_i \int e(t)dt + k_d \frac{d}{dt} e(t) \tag{19}$$

Note that k_p, k_i and k_d correspond to the PID controller gains and $e(t)$ represents the error between the reference and the state variable of interest. In the case of the reduced RWP model, the error is defined as $e(t) = r(t) - \Delta x_1$. Using the definition of the error and applying the Laplace transformation, the transfer function between the output Δx_1 and the reference signal is obtained as follows:

$$H(s) = \frac{\Delta X_1(s)}{R(s)} = \frac{-b(k_d s^2 + k_p s + k_i)}{s^3 - bk_d s^2 - (a + bk_p)s - bk_i} \tag{20}$$

The stability of the controller depends exclusively on the parameters of the PID, which implies that with these gains the poles of the system can be reallocated. Let us consider the following desired polynomial:

$$\begin{aligned} p_d(s) = \; & s^3 - (p_1 + p_2 + p_3)s^2 + \\ & (p_1p_2 + p_1p_3 + p_2p_3)s - p_1p_2p_3 \end{aligned} \tag{21}$$

where p_1, p_2 and p_3 are the desired poles of the closed-loop system, such that $p_1 \in \mathbb{R}^-$ and $(p_2, p_3) \in \mathbb{C} \wedge \mathrm{Re}(p_2, p_3) < 0 \wedge p_2 = conj(p_3)$.

After comparing the desired polynomial (21) with the characteristic polynomial in the transfer function given by (20), the PID gains can be obtained as

$$\begin{aligned} k_p = \; & -\tfrac{1}{b}(p_1p_2 + p_1p_3 + p_2p_3 + a) \\ k_i = \; & \tfrac{1}{b}p_1p_2p_3 \\ k_d = \; & \tfrac{1}{b}(p_1 + p_2 + p_3) \end{aligned}$$

It should be noted that the desired poles define the PID gains; nevertheless, to select the desired poles it is necessary to consider the operating limits of the control signal.

4.1.2. Approximate feedback linearization approach

Feedback linearization using Taylor's series is one of the most common control strategies for nonlinear systems around an operating point [19]. This approach tries basically to reallocate the open-loop eigenvalues so that the dynamical system becomes locally asymptotically stable [15].

Let us define the control input as follows:

$$\Delta u = -k_1 \Delta x_1 - k_2 \Delta x_2. \tag{22}$$

If the control input (22) is substituted into (18), the closed-loop dynamical system via approximate feedback linearization is obtained:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a + bk_1 & bk_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}. \tag{23}$$

The eigenvalues of the closed-loop dynamical system can be calculated by finding the roots of the characteristic polynomial:

$$\lambda^2 - bk_2\lambda - (a + bk_1) = 0. \tag{24}$$

If the desired polynomial is defined as $p_d(s) = \lambda^2 - (p_1 + p_2)\lambda + p_1p_2$ such that p_1 and p_2 are the desired poles of the closed-loop system defined by $\{(p_1, p_2) \in \mathbb{C} \wedge \mathrm{Re}(p_1, p_2) < 0\} \vee (p_1, p_2) \in \mathbb{R}^-$, it is then possible to obtain, by comparison with (24), the feedback gains presented below:

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$$k_1 = -\frac{1}{b}(p_1 p_2 + a) \wedge k_2 = \frac{1}{b}(p_1 + p_2) \tag{25}$$

Note that the feedback gains depend exclusively on the maximum values that the control input (22) can take.

4.2. Exact feedback linearization approach

In the case of nonlinear control, the exact feedback control theory is one of the most popular approaches. The main idea of this control technique is to obtain a nonlinear control law that allows transforming the nonlinear system into a linear equivalent system [19].

Let us consider the reduced RWP model defined by (12), and consider the following control law:

$$u = \frac{a}{b}\sin(x_1) - \frac{v}{b}. \tag{26}$$

where v represents the control law for the equivalent linear system. After substituting (26) in (12) and rearranging some terms, we obtain the following equivalent linear system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v. \tag{27}$$

For this equivalent linear system, it is easy to obtain an equivalent PID representation or pole reassignment, as presented in the two above subsections.

4.2.1. PID design

Defining the linear control input v as $v = k_p e(t) + k_i \int e(t)dt + k_d \frac{d}{dt}e(t)$, the transfer function between the pendulum angle x_1 and the reference value can be calculated:

$$H(s) = \frac{x_1(s)}{R(s)} = \frac{k_d s^2 + k_p s + k_i}{s^3 + k_d s^2 + k_p s + k_i} \tag{28}$$

Note that to control the equivalent linear system, it is necessary to find a third order polynomial with the same characteristics presented in (21), which produces the following PID gains:

$$\begin{aligned} k_p &= p_1 p_2 + p_1 p_3 + p_2 p_3 \\ k_i &= -p_1 p_2 p_3 \\ k_d &= -(p_1 + p_2 + p_3) \end{aligned}$$

4.2.2. Feedback design

For the equivalent linear system given by (27), it is possible to obtain an equivalent linear controller via feedback realization, by defining $v = -k_1 x_1 - k_2 x_2$. Substituting the linear control input v in (27) and rearranging some terms yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{29}$$

Note that the eigenvalues of (29) depend on the feedback gains k_1 and k_2. These eigenvalues can be calculated by $\lambda^2 + k_2\lambda + k_1 = 0$, which implies that the feedback gains may be obtained:

$$k_1 = p_1 p_2 \wedge k_2 = -p_1 - p_2 \tag{30}$$

4.3. Lyapunov control approach

The design of controllers using Lyapunov's direct method allows determining the mathematical structure of the control law that will guarantee stability in closed-loop for the dynamic system under study. In the specialized literature there is a strategy of control design called *Control Lyapunov Function* [18], which has strong mathematical foundations and is based on the analyses presented in Section 3 [8]. In this section we will propose two possible Lyapunov's functions to control the RWP system.

4.3.1. First function: stable controller design

Let us define the following Lyapunov candidate function:

$$\mathcal{V}(x) = \frac{1}{2}\left(k_1 x_1^2 + x_2^2\right) + 2a\sin^2\left(\frac{1}{2}x_1\right), \tag{31}$$

where k_1 represents a positive control feedback gain related to the x_1 state variable.

Note that (31) fulfills the first two conditions defined by (15), since (31) is a positive definite function with a global minimum located at the origin of the coordinates. To relate the control input to the output, the derivative with respect to time of Lyapunov's candidate function is given below:

$$\dot{\mathcal{V}}(x) = x_2(k_1 x_1 - bu + 2a\sin(x_1)). \tag{32}$$

To guarantee that (32) is negative semidefinite, as is required by the third condition of (15), we can select the control input u to be

$$u = \frac{1}{b}(k_1 x_1 + k_2 x_2 + 2a\sin(x_1)), \tag{33}$$

where $u \in \mathbb{R}$ and k_2 is a positive control feedback gain.

After substituting (33) in (32) and rearranging some terms, one obtains

$$\dot{\mathcal{V}}(x) = -k_2 x_2^2. \tag{34}$$

Recall that (34) is negative semidefinite, which implies (recurring to Theorem 1) that the dynamical system (4) is stable in the sense of Lyapunov.

By employing LaSalle's invariance principle [4], it is easy to prove, for the RWP system, that the origin of the coordinates is asymptotically stable in the sense of Lyapunov. If $\dot{\mathcal{V}}(x) = 0$, then $x_2 = 0$, which implies that $x_1 = 0$ (see the first equation of (4)), which confirms that the origin corresponds to the maximum invariant set and is asymptotically stable.

4.3.2. Second function: design of an asymptotically stable controller

In this section, we present an alternative Lyapunov candidate function that guarantees asymptotic stability without recurring to LaSalle's principle.

Let us consider the following Lyapunov candidate function, which fulfills the first two conditions of (15).

$$\mathcal{V}(x) = \frac{1}{2}(k_1 x_1 + k_2 x_2)^2. \tag{35}$$

Using (16), one obtains

$$\dot{\mathcal{V}}(x) = (k_1 x_1 + k_2 x_2)(k_1 x_2 + a k_2 \sin(x_1) - b k_2 u) \tag{36}$$

To guarantee that (36) is negative definite, we propose the following control input:

$$u = \frac{1}{b}\left(k_1 x_1 + k_2 x_2 + a\sin(x_1) + \frac{k_1}{k_2}x_2\right), \tag{37}$$

where $u \in \mathbb{R}$ and k_1 and k_2 are two positive feedback gains such that $k_2 \neq 0$.

After substituting the control input (37) into (36) and rearranging some terms, one obtains

$$\dot{\mathcal{V}}(x) = -k_2(k_1 x_1 + k_2 x_2)^2 \tag{38}$$

Note that (8) is negative definite, which implies that the RWP system (4) is asymptotically stable at the origin.

4.4. Additional comments

In the design of controllers using Lyapunov's direct method, the existence of at least one Lyapunov candidate function is required. In the case of the RWP system, we have presented two different quadratic functions; nevertheless, we claim that for the RWP

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reduced model (4) there are multiple Lyapunov's candidate functions that guarantee stability or asymptotic stability in closed-loop.

On the other hand, one of the biggest advantages of Lyapunov's direct method is the possibility of obtaining nonlinear controllers directly in two steps, without recurring to the eigenvalue analysis needed for the proportional integral and linear state variables feedback control approaches.

5. Simulation results

This section presents a comparison of the performance of the RWP for the different controllers presented in Section 4. For the sake of making a fair comparison and obtain an equal dynamic response for all controllers, the poles in closed-loop are located in the same place:

$$p_1 = 0 \quad p_2 = -35 \quad p_3 = -100.$$

The parameters of the RWP system and of each controller are given in Tables 1 and 2, respectively. Three different tasks are considered. For each of them, x_1, x_2 and u are shown. The initial conditions for all tasks are $x_1 = 0.12$ rad and $x_2 = 0$ rad/s, and also the limits of the control signal are assumed to be between -10 and 10.

5.1. Task 1: the normal case

Fig. 3 illustrates the response of the RWP under all controllers. It should be noted that only one response is shown in Fig. 3 since all the employed controllers have the same response dynamics. This is because from the design of the controllers, all them are adjusted to have the same response dynamics. However, the only controller that guarantees asymptotic stability is the one presented in Section 4.3.2.

Fig. 4 presents the phase plane trajectory of the RWP for task 1, which moves to the origin from the initial conditions, demonstrating that the system is stable.

5.2. Task 2: uncertainty

The uncertainty in the RWP can be expressed by [10,6]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ a\sin(x_1) \end{bmatrix} + \Delta A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \end{bmatrix} u$$

where ΔA is the matrix that represents the uncertainty of the RWP, which includes the uncertainties in the parameters. These uncertainties can make it difficult or impossible to get a precise measurement, as the parameters tend to vary as a function of time, of the temperature or unmodeled dynamics, among others. When ΔA is added, this means that the parameters of the RWP are changed, as determined by the following equation:

$$\Delta A = \begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix}$$

Fig. 5 shows the response of the RWP for task 2. In this task, all the controllers can take the RWP to the equilibrium point without any problem even if there are uncertainties in the system.

Comparing Figs. 3 and 5 it can be seen that in task 2, when there are added uncertainties, the response of the RWP has a greater oscillation and a slower response time. In this task as well, it can be seen that the system is stable since it is also able to move from the initial conditions to the origin, as shown in Fig. 6.

5.3. Task 3: external disturbances

The response of the RWP using all proposed controllers after adding an angle disturbance with a value equal to 1 rad at time 0.5 s is shown in Fig. 7.

Table 1
RWP Parameters [2].

a	b	c
$78.4\ \left(\frac{rad}{s}\right)^2$	$1.08\ \frac{rad}{s^2}$	$198\frac{rad}{s^2}$

Table 2
Controller Parameters.

Controller	Parameters		
Section 4.1.1	$k_p = -3313.3$	$k_i = 0$	$k_d = -125$
Section 4.1.2	$k_1 = -3313.3$		$k_2 = -125$
Section 4.2.1	$k_p = 3500$	$k_i = 0$	$k_d = 135$
Section 4.2.2	$k_1 = 3500$		$k_2 = 135$
Section 4.3.1	$k_1 = 3500$		$k_2 = 135$
Section 4.3.2	$k_1 = 3500$		$k_2 = 35$

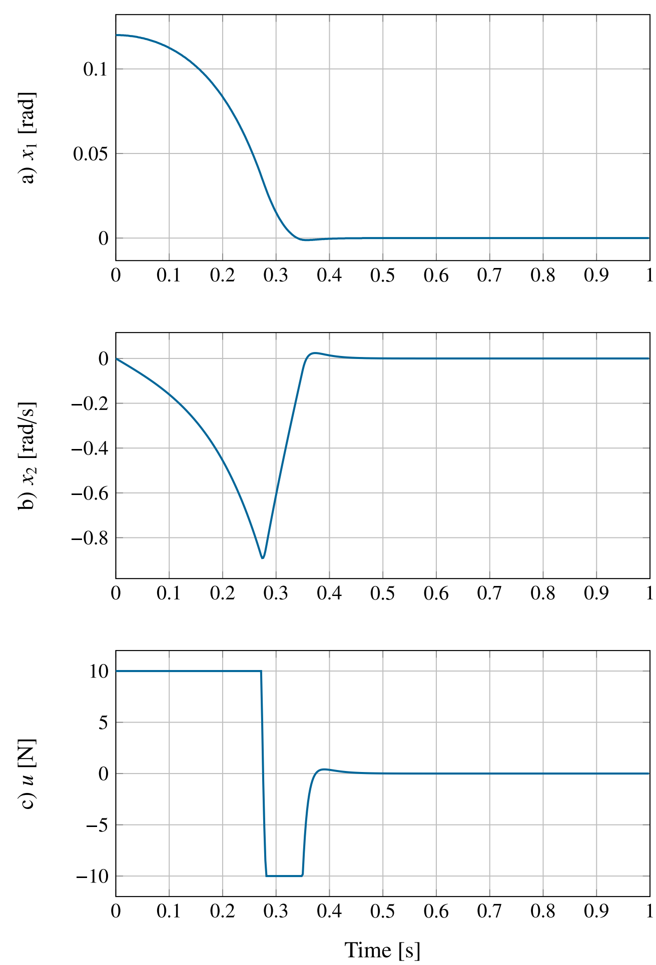

Fig. 3. Response of the RWP for task 1: Pendulum angle, b) Angular velocity, and c) Control signal.

The response of the RWP for the all employed controllers are significantly faster and the oscillation of the pendulum angle with respect to the equilibrium point does not much change (see Fig. 7a). Fig. 8 shows the phase plane trajectory of the RWP system under an external disturbance.

It can be seen in Fig. 8 that the phase plane trajectory of the RWP system leaves the origin and moves back to the origin after adding the angle disturbance, which indicates that the system is stable for all the controllers used.

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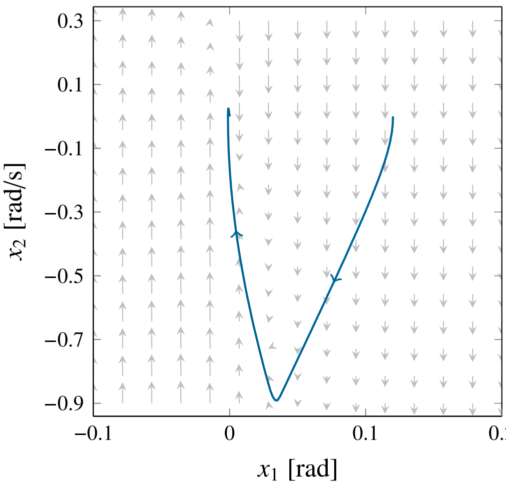

Fig. 4. Phase plane trajectory of the RWP for task 1.

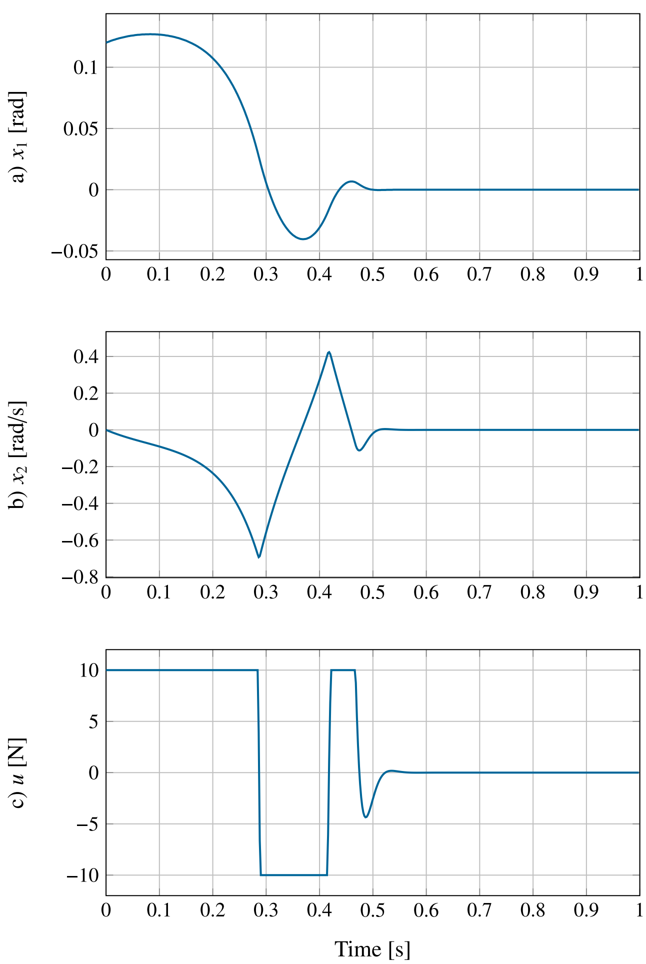

Fig. 5. Response of the RWP for task 2: Pendulum angle, b) Angular velocity, and c) Control signal.

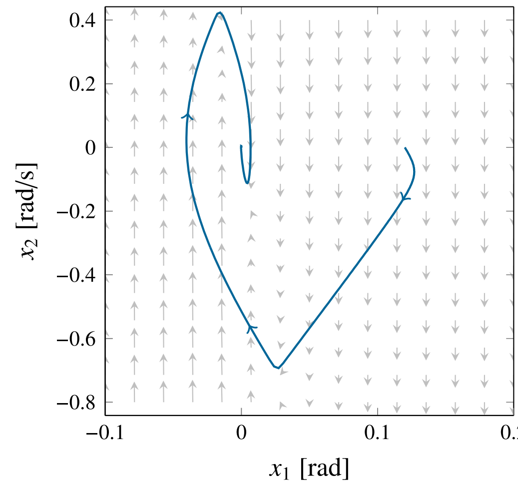

Fig. 6. Phase plane trajectory of the RWP for task 2.

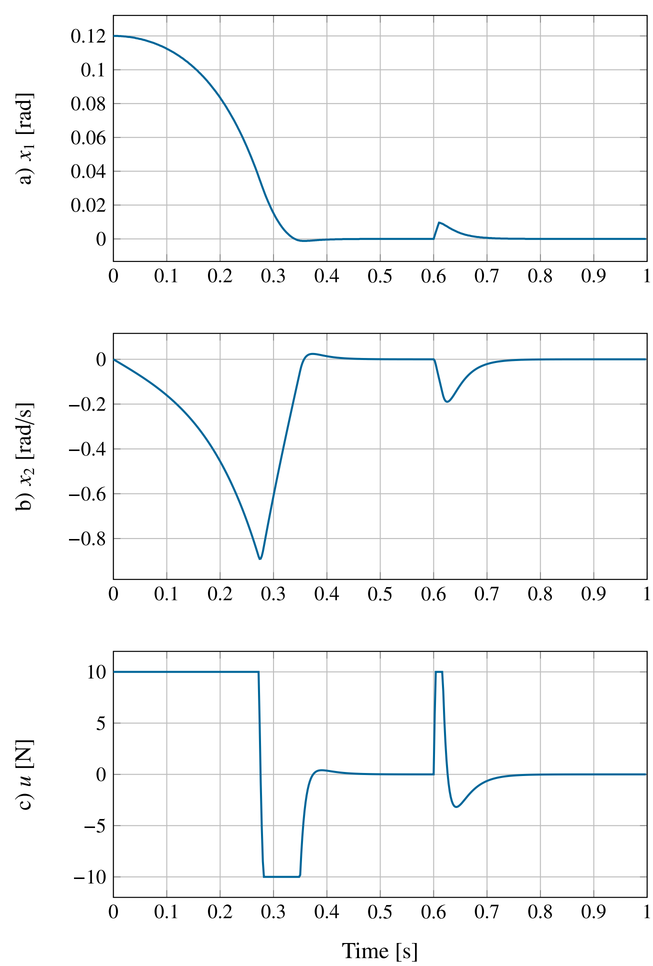

Fig. 7. Response of the RWP for task 3: Pendulum angle, b) Angular velocity, and c) Control signal.

5.4. Additional comments

A and B present the design of a swing-up control strategy for bringing the pendulum bar from its natural position of equilibrium (vertical inferior position) closer to the vertical upright position. For doing this task, an energy-based approach is presented in [14] by using a soft switching method. In addition, multiple simulations are included to compare all six studied controllers in order to prove that all of them have pretty similar dynamic performances when the control gains are adjusted adequately.

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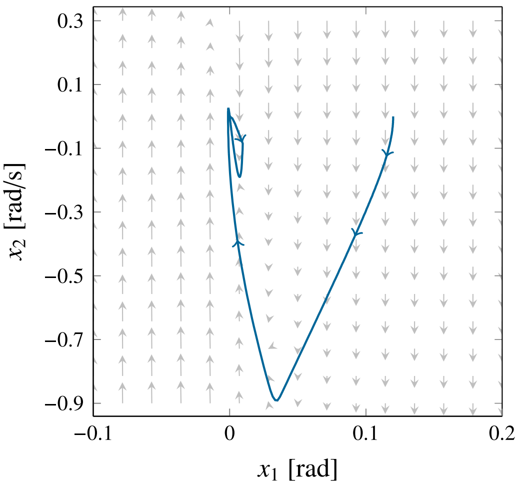

Fig. 8. Phase plane trajectory of the RWP for task 3.

6. Conclusions

In this paper, different controllers were presented to control a reaction wheel pendulum. All employed controllers have been tested by using three simulation tasks: the typical case, including uncertainties, and in response to external disturbances. All proposed controllers had the same dynamical behavior since equivalent controller gains were selected for their design.

The design of the controllers using exact feedback linearization or Lyapunov's direct method guarantees stability or asymptotic stability, which is the main advantage of these techniques in comparison with the classical feedback and proportional integral controllers based on Taylor series linearizations.

The reaction wheel pendulum mechanism is a complicated nonlinear dynamical system that allows designing multiple control strategies with excellent dynamical performances. For this reason, the RWP continues to be an important system in dynamic analysis and control theory, and affords an introduction to research involving complicated dynamical models in an easy and simple form.

Appendix A. Swing-up strategy for the pendulum

To bring the pendulum closer to the equilibrium position, we employed an energy-based method, as proposed in [14]. This strategy works with the total potential energy stored in the pendulum bar [20,19][1]. The potential energy stored in the pendulum can be calculated as

$$W_p = \frac{1}{2}J_p(\dot{\varphi}^2) + mgl(\cos(\varphi) - 1), \tag{A.1}$$

where J_p denotes the inertia of the pendulum, l and m are its length and mass, while g is the acceleration due to gravity.

If we take the time derivative of the potential energy W_p, then

$$\dot{W}_p = \dot{\varphi}\left(J_p\ddot{\varphi} - mgl\sin(\varphi)\right). \tag{A.2}$$

Now, if (A.2) is compared with (1), then it is possible to find a direct relation between the variation of the potential energy and the control input:

$$\dot{W}_p = -J_p b\dot{\varphi}u. \tag{A.3}$$

Note that the potential energy given by (A.1) has a global minimum ($W_p = -2mgl$) when the pendulum is located at its natural equilibrium point (vertical bottom position). In addition, the potential energy will be zero at the vertical upper position. In order to bring this energy to zero, all that is needed is to ensure that $\dot{W}_p > 0$ [14], which implies that the control input can be selected to be

$$u = -k_u sign(\dot{\varphi}). \tag{A.4}$$

Note that (12) uses k_u as a positive constant to manipulate the required time to reach a vertical position; in addition, it uses the sign of the angular speed to reinforce the movement of the pendulum in order to increase the amplitude of its oscillations [19] by applying a torque in the same direction as the movement.

Finally, to ensure a soft movement of the pendulum, the control strategy is divided into three possibilities:

$$u = \begin{cases} \text{Eq.} \quad (A.4), & if \ \frac{5\pi}{6} \leqslant \varphi \leqslant \frac{5\pi}{6} \\ 0, & \text{otherwise} \end{cases} \tag{A.5}$$

Appendix B. Additional simulations

In order to compare all dynamic performances of the studied linear and nonlinear controllers, we tested all of them by considering the following initial conditions and parameters: $x_1 = \pi, x_2 = 0$ and $k_u = 10$.

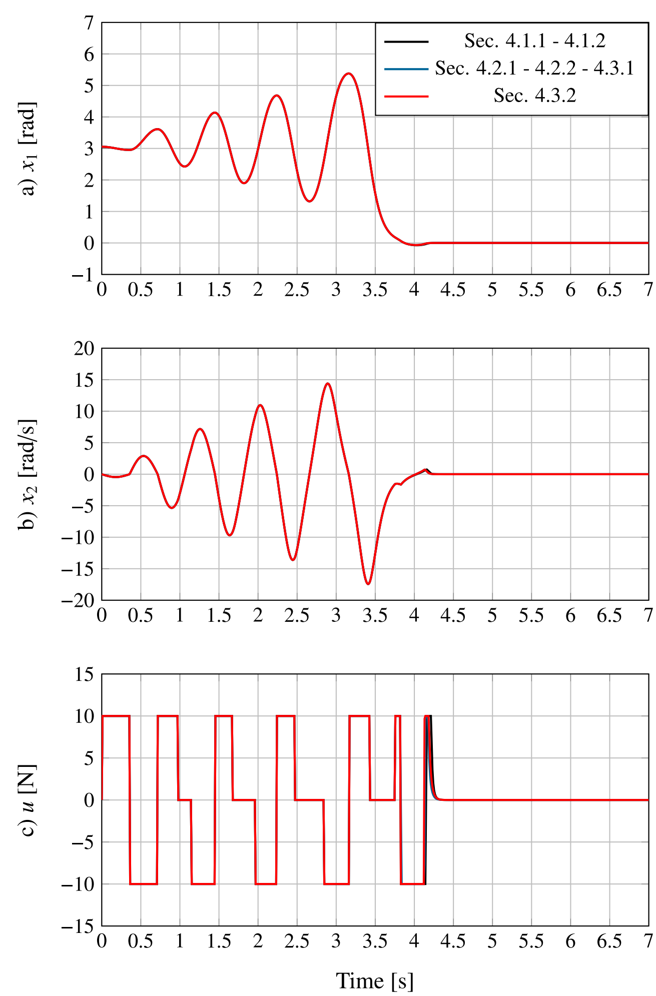

Fig. B.9. Dynamic behavior of the reduced RWP model (4) for all the proposed controllers: a) Angular position, b) Angular velocity and c) Control signal.

[1] Note that the kinetic energy of the reaction wheel is not taken into account, since it does not influence its angular position [14].

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Fig. B.9 depicts all dynamic performances for the controllers studied in Section 4 when they are integrated with the soft switching control strategy defined by (A.5).

Note that all studied controllers produce overlapping between angular positions (also in angular velocities) (see Figs. B.9a) and B.9b)) by minimal changes at the control inputs (see Fig. B.9c)). In addition, it is possible to observe that the control input defined by (A.4) allows reaching a region closer to the desired operating point (vertical upper position) from this natural vertical inferior position when 3.5 s have passed. To do so, the control input produces a square wave function in order to increase the oscillation of the pendulum; furthermore, in some periods of time, this control input takes zero values to allow free movement of the pendulum bar and reduce its velocity, for the sake of minimizing the effort produced by the commutation when the controller proposed in Section 4 is used.

Note that the main idea of plotting all controllers (included the swing up strategy) in Fig. B.9 is to show that with an adequate calibration of each controller it is possible to obtain pretty similar dynamical performances, which effectively corresponds to the main purpose of this tutorial paper in reviewing linear and nonlinear controllers from the point of view of Lyapunov's stability theory.

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