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RESEARCH ARTICLE

On some families of subsemigroups of a numerical semigroup

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Abstract

To a given numerical semigroup S we associate a family of subsemigroups $\{\partial^n S\}$, $n \in \mathbb{N}$, that permits us to understand some of the structure of S. We characterize this family in case S is a supersymmetric numerical semigroup or S has maximal embedding dimension. We also prove some properties related to embedding dimension and certain symmetry of the minimal generating set of the members of this family.

Keywords Numerical semigroup · Supersymmetric · Maximal embedding dimension · Minimal generating set

1 Introduction

A *numerical semigroup* is a subset S of the set of natural numbers $\mathbb{N}$ such that S is closed under the sum, $0 \in S$, and $\mathbb{N}\backslash S$ is a finite set. If S is a numerical semigroup, then S is finitely generated, that is, there are $a_1, a_2, \dots, a_r \in S$ such that every element in S can be written in the form $\sum_{i=1}^{r} c_i a_i$ where $c_i \in \mathbb{N}$, $i = 1, 2, \dots, r$. The subset $S^*\backslash(S^* + S^*)$ of S has the properties that it is finite, generates S and every generating set of S contains it. We call the set $S^*\backslash(S^* + S^*)$ *the minimal generating set of* S, and we will denote it by $\beta(S)$. The cardinality of $\beta(S)$ is called the *embedding dimension of* S and it is denoted by $e(S)$; the least positive integer belonging to S is called the *multiplicity of*

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S, and it is denoted by $m(S)$. It is known that $m(S) = \min \beta(S)$ and $e(S) \leq m(S)$. When $e(S) = m(S)$, we say that *S has maximal embedding dimension*. If $S \neq \mathbb{N}$, the maximum element in $\mathbb{N} \backslash S$ is called the *Frobenius number of S*, and it is denoted by $F(S)$; and the cardinality of $\mathbb{N} \backslash S$ is the *genus of S*, which is denoted by $g(S)$. If $n \in S \backslash \{0\}$, the *Apéry set of n in S*, denoted by $A(S; n)$, is defined as follows

$$A(S;n) = \{s \in S : s - n \notin S\}.$$

The Apéry set of n in S has the following properties (see [1, 3]):

1. $|A(S;n)| = n$.
2. Every element in S can be written uniquely in the form $an + w$, where $a \in \mathbb{N}$ and $w \in A(S;n)$. Thus, the set $\left[A(S;n) \backslash \{0\}\right] \cup \{n\}$ generates S.

Now we introduce some terminology in order to understand the main results of this paper. For a numerical semigroup S, the elements of $\beta(S)$ are not expressible as a sum of two nonzero elements of S, but any other nonzero element of S can be represented as a sum of at least two elements in $\beta(S)$. Note that the set $S \backslash \beta(S)$ is a numerical semigroup contained in S. We denote this numerical semigroup by ∂S. We observe that the elements in $\beta(\partial S)$ are precisely the nonzero elements in S that are expressible as a sum of at least two nonzero elements of S, but that are not a sum of two elements in ∂S. Note that the elements in $\beta(\partial S)$ cannot be written as a sum of 4 nonzero elements of S, but they can be written as a sum of 2 or 3 nonzero elements of S. We can consider the numerical semigroup $\partial^2 S := \partial S \backslash \beta(\partial S)$, its minimal generating set $\beta(\partial^2 S)$ and note that elements in $\beta(\partial^2 S)$ cannot be expressed as a sum of 8 nonzero elements of S nor as a sum of less than 4 nonzero elements of S. Actually, we define recursively a family of numerical semigroups $\{\partial^n S\}_{n \in \mathbb{N}}$, as follows:

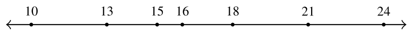
1. $\partial^0 S = S$, and
2. $\partial^{n+1} S = \partial(\partial^n S)$, for $n \in \mathbb{N}$.

This family of numerical semigroups $\{\partial^n S\}_{n \in \mathbb{N}}$ and the family of minimal generating sets $\{\beta(\partial^n S)\}_{n \in \mathbb{N}}$ can help us to understand some of the structure of the numerical semigroup S. We see that the set $\beta(\partial^n S)$ is formed by those nonzero elements of S that are not expressible as a sum of fewer than 2^n nonzero elements of S nor as a sum of at least 2^{n+1} nonzero elements of S. However, the semigroup $\partial^n S$ acquires properties that S may not have, as n increases. We explain precisely what this means. Let $\mathcal{P}$ be a property of numerical semigroups. We will say that the property $\mathcal{P}$ *eventually appears in S* if there exists $n_0 \in \mathbb{N}$ such that $\partial^n S$ has the property $\mathcal{P}$, for all $n \geq n_0$. In general, one would like to prove that for a given property $\mathcal{P}$ of numerical semigroups and any numerical semigroup S, the property $\mathcal{P}$ eventually appears in S, but this depends on the property $\mathcal{P}$ and the numerical semigroup S, as we will see later.

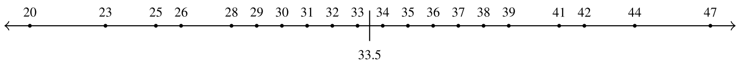
In many examples, the set $\beta(\partial^n S)$ presents a nice symmetry property as n increases. For instance, consider the numerical semigroup $S = \langle 5, 8 \rangle$; then we have

$$\beta(\partial S) = \{10, 13, 15, 16, 18, 21, 24\}.$$

We plot this numbers in the line as follows:



Next, we compute $\beta(\partial^2 S) = \{20, 23, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 44, 47\}$ and we plot these numbers in the line:



We observe that the elements of $\beta(\partial^2 S)$ are placed symmetrically with respect to $(47+20)/2 = 33.5$. This symmetry property of the set $\beta(\partial^2 S)$ can be formulated as follows: for any pair of positive integers a and b, if $a + b = \min \beta(\partial^2 S) + \max \beta(\partial^2 S)$, then $a \in \beta(\partial^2 S)$ if and only if $b \in \beta(\partial^2 S)$.

We call a numerical semigroup S *β-symmetric* if it satisfies that for any pair of positive integers a and b, if $a + b = \min \beta(S) + \max \beta(S)$, then $a \in \beta(S)$ if and only if $b \in \beta(S)$. This condition means precisely that when we put the elements of $\beta(S)$ in the line, they are placed symmetrically with respect to $(\min \beta(S) + \max \beta(S))/2$.

A numerical semigroup S minimally generated by $a_1, a_2, \dots, a_r$ is *supersymmetric* (see [2]) if and only if there are pairwise relatively prime numbers $u_1, \dots, u_r$ such that

$$a_i = \prod_{k=1, k\neq i}^{r} u_k$$

for $i = 1, 2, \dots, r$. For instance, all numerical semigroups with embedding dimension 2 are supersymmetric.

Now we establish the main results of this paper. We prove that if S, minimally generated by $a_1, a_2, \dots, a_r$, is supersymmetric (assuming $u_1 > u_j$ for $j = 2, \dots, r$), then

$$\partial^n S = \left\{ \sum_{i=1}^{r} a_i x_i \,:\, x_i \in \mathbb{N}, \sum_{i=1}^{r} x_i \geq 2^n \right\} \cup \{0\}$$

and

$$\beta(\partial^n S) = \left\{ \sum_{i=1}^{r} a_i x_i \,:\, 2^n \leq \sum_{i=1}^{r} x_i < 2^{n+1}, 0 \leq x_i < u_i, i = 2, \dots, r \right\}$$

for all $n \geq 0$. We also prove that if S is supersymmetric, then the properties of having maximal embedding dimension and β-symmetry, eventually appear in S.

We prove that if S has maximal embedding dimension, then ∂S also has this property. We have conjectured that the property of having maximal embedding dimension eventually appears in S. Evidence in examples suggests that if S has not maximal embedding dimension, then

$$e(\partial S) \geq 2e(S) + 1. \tag{1}$$

We prove that (1) is true if S is supersymmetric or S has embedding dimension 3.

2 General properties of the family $\{\partial^n S\}_n$

We can give a description of the sets $\beta(\partial^n S)$ in terms of the length of representations of elements as sums of nonzero elements of S. For $s \in S\backslash\{0\}$, let $l(s)$ and $L(s)$ be the least and greatest number of summands among all representations of s as a sum of nonzero elements of S, respectively. Then, we have

$$\beta(\partial^n S) = \{s \in S\backslash\{0\} \,:\, 2^n \leq l(s), L(s) < 2^{n+1}\}.$$

But this description of $\beta(\partial^n S)$ is not useful when we work with a specific semigroup S.

We prove now that the family of subsets of S, $\{\beta(\partial^n S)\}_{n\in\mathbb{N}}$, is a partition of $S\backslash\{0\}$. We need the following lemma.

Lemma 1 *If S is a numerical semigroup and $n \geq 1$, then*

$$\partial^n S = S\backslash \bigcup_{i=0}^{n-1} \beta(\partial^i S). \tag{2}$$

Proof By induction on n. By definition, $\partial S = S\backslash\beta(S) = S\backslash\beta(\partial^0 S)$. Assuming that $\partial^n S = S\backslash \bigcup_{i=0}^{n-1} \beta(\partial^i S)$, where $n \geq 1$, we have

$$\begin{aligned}\partial^{n+1} S &= \partial(\partial^n S)\\ &= \partial^n S\backslash\beta(\partial^n S)\\ &= \left[S\backslash \bigcup_{i=0}^{n-1} \beta(\partial^i S)\right]\backslash\beta(\partial^n S)\\ &= S\backslash \bigcup_{i=0}^{n} \beta(\partial^i S).\end{aligned}$$

□

If S is a numerical semigroup and $m(S)$ is its multiplicity, then $m(\partial S) = 2m(S)$, since the minimal nonzero element in S that does not belong to $\beta(S)$ is $2m(S)$. So, by induction on n, we have

$$m(\partial^n S) = 2^n m(S) \tag{3}$$

for all $n \in \mathbb{N}$.

Proposition 1 *Let S be a numerical semigroup. Then the sets $\beta(\partial^n S)$, for $n \geq 0$, form a partition of $S \backslash \{0\}$.*

Proof If $m < n$, where $n, m \geq 0$, then $n \geq 1$ and by (2), we have $\partial^n S = S \backslash \bigcup_{i=0}^{n-1} \beta(\partial^i S)$, so that $\beta(\partial^n S) \cap \beta(\partial^m S) = \varnothing$. Now, in order to prove that $S \backslash \{0\} = \bigcup_{n \in \mathbb{N}} \beta(\partial^n S)$, note that by (3) there is an increasing sequence

$$m(S) < m(\partial S) < \cdots < m(\partial^n S) < \cdots$$

If $s \in S \backslash \{0\}$, then there exists $n_0 \geq 1$ such that $s < m(\partial^{n_0} S)$, and this implies that $s \notin \partial^{n_0} S$. So, by (2), $s \in \beta(\partial^i S)$ for some $i < n_0$. This ends the proof. □

Proposition 2 *Let S be a numerical semigroup. Then*

(1) $F(\partial^n S) = \max\{F(S), \max \beta(\partial^{n-1} S)\}$, *for all $n \geq 1$.*
(2) $g(\partial^n S) = g(S) + e(S) + e(\partial S) + \cdots + e(\partial^{n-1} S)$, *for all $n \geq 1$.*

Proof

(1) If $\max \beta(\partial^{n-1} S) \leq F(S)$, the Frobenius number of $\partial^n S$ is $F(S)$. If $\max \beta(\partial^{n-1} S) > F(S)$, then the Frobenius number of $\partial^n S$ is $\max \beta(\partial^{n-1} S)$. In any case, we have $F(\partial^n S) = \max\{F(S), \max \beta(\partial^{n-1} S)\}$.
(2) Using (2) we have

$$\begin{aligned} g(\partial^n S) &= |\mathbb{N} \backslash \partial^n S| \\ &= \left| \mathbb{N} \backslash \left(S \backslash \bigcup_{i=0}^{n-1} \beta(\partial^i S) \right) \right| \\ &= \left| (\mathbb{N} \backslash S) \cup \bigcup_{i=0}^{n-1} \beta(\partial^i S) \right| \\ &= |\mathbb{N} \backslash S| + \sum_{i=0}^{n-1} |\beta(\partial^i S)| \\ &= g(S) + \sum_{i=0}^{n-1} e(\partial^i S) \end{aligned}$$

□

By part (1) of Proposition 2, for any numerical semigroup S, there is some $n_0 \geq 1$ such that $F(\partial^n S) = \max \beta(\partial^{n-1} S)$, for all $n \geq n_0$. Thus, the property that $F(\partial S) = \max \beta(S)$ eventually appears in S.

Theorem 1 *Let S be a numerical semigroup with maximal embedding dimension. Then, ∂S also has maximal embedding dimension.*

Proof We must prove that $e(\partial S) = 2m(S)$. Let $n = e(S)$. Since S has maximal embedding dimension, we also have $n = m(S)$. As $\beta(S)$ is a subset of $[A(S;n)\backslash\{0\}] \cup \{n\}$ and $|\beta(S)| = n = \big|[A(S;n)\backslash\{0\}] \cup \{n\}\big|$, we have $\beta(S) = [A(S;n)\backslash\{0\}] \cup \{n\}$.

Now, every element of S can be written uniquely as $an + w$, where $a \in \mathbb{N}$ and $w \in A(S;n)$. Nonzero elements in ∂S can be written in the form $an + w$ where $a \geq 2$ or, $a = 1$ and $w \neq 0$. Thus, the set of elements of the form $an + w$, where $a \in \{1, 2\}$ and $w \in \beta(S)$, generates ∂S.

We claim that $\beta(\partial S) = \{an + w : a \in \{1, 2\}, w \in \beta(S)\}$. It suffices to show that the sum of two elements in $\{an + w : a \in \{1, 2\}, w \in \beta(S)\}$ does not lie in $\{an + w : a \in \{1, 2\}, w \in \beta(S)\}$. Now, if $a_1n + w_1 = (a_2n + w_2) + (a_3n + w_3)$, where $a_i \in \{1, 2\}$ and $w_i \in \beta(S)$, $i = 1, 2, 3$; then $w_1 = (a_2 + a_3 - a_1)n + w_2 + w_3$ (note that $a_2 + a_3 - a_1 \geq 0$); but this means that $w_1 \notin \beta(S)$, a contradiction. This proves our claim.

Finally, there are $2n$ elements of the form $an + w$, where $a \in \{1, 2\}$ and $w \in \beta(S)$. This ends the proof. □

As we can see in the proof of Theorem 1, if S has maximal embedding dimension, then

$$\beta(\partial S) = \{am(S) + w : 1 \leq a \leq 2, w \in \beta(S)\}.$$

By induction, we get the following result.

Proposition 3 *Let S be a numerical semigroup with maximal embedding dimension. If $n \geq 0$, then*

$$\beta(\partial^n S) = \{am(S) + w : 2^n - 1 \leq a \leq 2^{n+1} - 2, w \in \beta(S)\}.$$

Hence, $\partial^n S = \{am(S) + w : 2^n - 1 \leq a, w \in \beta(S)\} \cup \{0\}$.

Given a numerical semigroup S, we wish to prove that the property of having maximal embedding dimension eventually appears in S. In fact, all evidence suggests that this is true. We have the following conjecture.

Conjecture 1 *For any numerical semigroup S, the property of having maximal embedding dimension eventually appears in S.*

Of course, if we are able to prove that there is some $n_0 \in \mathbb{N}$ such that $\partial^{n_0} S$ has maximal embedding dimension, then, by Theorem 1, the property of having maximal embedding dimension eventually appears in S. Evidence shows that $e(\partial^n S)$ strictly increases with n, and that if S has not maximal embedding dimension, then $e(\partial S) \geq 2e(S) + 1$.

Conjecture 2 *If S has not maximal embedding dimension, then*

$$e(\partial S) \geq 2e(S) + 1.$$

We will prove later that this conjecture is true in the cases of S supersymmetric or $e(S) = 3$.

Proposition 4 *Let us assume that S has maximal embedding dimension. If S is β-symmetric, then ∂S is β-symmetric.*

Proof Let S be maximally generated by $n = a_1 < a_2 < \cdots < a_n$. Being β-symmetric is equivalent to say that for any $j \in \{1, 2, \dots, n\}$, $a_1 + a_n - a_j = a_k$ for some $k \in \{1, 2, \dots, n\}$. Now, by Proposition 3, $\beta(S) = \{ra_1 + a_j : r \in \{1, 2\}, 1 \le j \le n\}$. Note that $\max \beta(\partial S) = 2a_1 + a_n$ and $\min \beta(\partial S) = 2a_1$. Thus, to prove that ∂S is β-symmetric, we have to prove that $(4a_1 + a_n) - (ra_1 + a_j) \in \beta(\partial S)$, where $r \in \{1, 2\}$ and $1 \le j \le n$. If fact, if $r \in \{1, 2\}$ and $1 \le j \le n$, then $(4a_1 + a_n) - (ra_1 + a_j) = (3 - r)a_1 + (a_1 + a_n - a_j) \in \beta(\partial S)$, since $3 - r \in \{1, 2\}$ and $a_1 + a_n - a_j = a_k$ for some $1 \le k \le n$. This ends the proof. □

If $0 < r \le k$, let $S_{k,r} = \{0, k, k + r, \rightarrow\}$. Then, $\partial^n S_{k,r} = S_{2^n k,r}$ for all $n \ge 0$. We have

$$\beta(S_{k,r}) = \{k\} \cup (\{k + r, k + r + 1, \dots, 2k + r - 1\} \setminus \{2k\}),$$

so we see that $S_{k,r}$ is not β-symmetric for any $k > 1$ (also, $S_{k,r}$ has maximal embedding dimension). Thus, the property of being β-symmetric does not appear eventually in $S_{k,r}$.

3 Supersymmetric numerical semigroups

In this section, $u_1, u_2, \dots, u_r$ are integers greater than 1 that are pairwise relatively prime and $u_1 > u_j$ for $j = 2, \dots, r$. For $i = 1, 2, \dots, r$, let

$$a_i = \prod_{k=1, k \neq i}^{r} u_k.$$

Lemma 2 *The integer solutions of the linear equation*

$$\sum_{i=1}^{r} a_i x_i = 0 \tag{4}$$

are of the form $x_i = u_i \alpha_i$, where α_i is an integer for $i = 1, 2, \dots, r$, and $\sum_{i=1}^{r} \alpha_i = 0$.

Proof If $x_1, x_2, \dots, x_r$ satisfy (4), then $u_i = \gcd(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$ divides x_i; so $x_i = u_i \alpha_i$ for some integer α_i. Therefore, by replacing x_i into (4) we get

$$\sum_{i=1}^{r} a_i x_i = \sum_{i=1}^{r} a_i(u_i \alpha_i)$$
$$= \sum_{i=1}^{r} (u_1 u_2 \cdots u_r)\alpha_i.$$

Thus, $\sum_{i=1}^{r} \alpha_i = 0$. The converse is easy to verify. □

Lemma 3 *If* $\sum_{i=1}^{r} a_i x_i = \sum_{i=1}^{r} a_i x_i'$, *where* $x_i, x_i' \in \{0, 1, \dots, u_i - 1\}$ *for* $i = 2, \dots, r$, *then* $x_i = x_i'$ *for all* $i \in \{1, 2, \dots, r\}$.

Proof Assume that $\sum_{i=1}^{r} a_i x_i = \sum_{i=1}^{r} a_i x_i'$, where $0 \le x_i < u_i$ and $0 \le x_i' < u_i$ for $i = 2, \dots, r$. By Lemma 2, for $i = 2, \dots, r$, we have $u_i \mid (x_i - x_i')$, and since $0 \le x_i < u_i$ and $0 \le x_i' < u_i$, it follows that $x_i = x_i'$, $i = 2, \dots, r$. Then $a_1 x_1 + \sum_{i=2}^{r} a_i x_i = a_1 x_1' + \sum_{i=1}^{r} a_i x_i$, which implies that $a_1 x_1 = a_1 x_1'$, and therefore $x_1 = x_1'$. This ends the proof. □

Let S be the numerical semigroup generated by $a_1, a_2, \dots, a_r$. Of course, we have $\beta(S) = \{a_1, a_2, \dots, a_r\}$.

Lemma 4 *Every element in S can be represented in a unique way in the form*

$$\sum_{i=1}^{r} a_i x_i,$$

where $0 \le x_i < u_i$, *for* $i = 2, 3, \dots, r$.

Proof Every element in S is of the form $\sum_{i=1}^{r} a_i y_i$ where $y_i \ge 0$, $i = 1, 2, \dots, r$. For $i = 2, \dots, r$, there are non-negative integers q_i and x_i such that $y_i = u_i q_i + x_i$ and $0 \le x_i < u_i$. Then

$$\sum_{i=1}^{r} a_i y_i = a_1 y_1 + \sum_{i=2}^{r} a_i(u_i q_i + x_i)$$
$$= a_1 y_1 + \sum_{i=2}^{r} a_i u_i q_i + \sum_{i=2}^{r} a_i x_i$$
$$= a_1 y_1 + \sum_{i=2}^{r} (u_1 u_2 \cdots u_r) q_i + \sum_{i=2}^{r} a_i x_i$$
$$= a_1 \left[y_1 + \sum_{i=2}^{r} u_1 q_i \right] + \sum_{i=2}^{r} a_i x_i$$
$$= \sum_{i=1}^{r} a_i x_i,$$

where $x_1 = y_1 + \sum_{i=2}^{r} u_1 q_i$. This shows that every element in S can be expressed in the desired way. The uniqueness follows from Lemma 3. □

Theorem 2 *Let $S = \langle a_1, \ldots, a_r \rangle$. Then*

$$\partial^n S = \left\{ \sum_{i=1}^{r} a_i x_i \,:\, x_i \geq 0, \sum_{i=1}^{r} x_i \geq 2^n \right\} \cup \{0\}$$

and

$$\beta(\partial^n S) = \left\{ \sum_{i=1}^{r} a_i x_i \,:\, 2^n \leq \sum_{i=1}^{r} x_i < 2^{n+1}, 0 \leq x_i < u_i, i = 2, \ldots, r \right\}$$

for all $n \geq 0$.

Proof For each $n \geq 0$, we define

$$S_n := \left\{ \sum_{i=1}^{r} a_i x_i \,:\, x_i \geq 0, \sum_{i=1}^{r} x_i \geq 2^n \right\} \cup \{0\}$$

and

$$T_n := \left\{ \sum_{i=1}^{r} a_i x_i \,:\, 2^n \leq \sum_{i=1}^{r} x_i < 2^{n+1}, 0 \leq x_i < u_i, i = 2, \ldots, r \right\}.$$

It is clear that S_n is a numerical semigroup. We show, by induction on n, that $\partial^n S = S_n$ and $\beta(\partial^n S) = T_n$, for all $n \geq 0$.

First, we have $S = \partial^0 S = S_0$. Besides, the condition $2^0 = 1 \leq \sum_{i=1}^{r} x_i < 2$ means that $\sum_{i=1}^{r} x_i = 1$, which is equivalent to say that one of the x_i is 1 and the other are 0. Thus, $T_0 = \{a_1, a_2, \ldots, a_r\} = \beta(S) = \beta(\partial^0 S)$.

We assume that $\partial^k S = S_k$ and $\beta(\partial^k S) = T_k$ for $k = 0, \ldots, n$. Now,

$$\partial^{n+1} S = S \backslash \bigcup_{k=0}^{n} \beta(\partial^k S) = S \backslash \bigcup_{k=0}^{n} T_k$$

and observe that

$$\bigcup_{k=0}^{n} T_k = \left\{ \sum_{i=1}^{r} a_i x_i \,:\, 0 \leq x_i < u_i, i = 2, \ldots, r, \text{ and } 1 \leq \sum_{i=1}^{r} x_i < 2^{n+1} \right\}.$$

To show that $S_{n+1} \subseteq \partial^{n+1} S$, we take $x \in S_{n+1}$ and write $x = \sum_{i=1}^{r} a_i x_i$ where $\sum_{i=1}^{r} x_i = 0$ or $\sum_{i=1}^{r} x_i \geq 2^{n+1}$. If $\sum_{i=1}^{r} x_i = 0$, then $x = 0 \in \partial^{n+1} S$. Now, we assume that $\sum_{i=1}^{r} x_i \geq 2^{n+1}$. We have to show that $x \notin \bigcup_{k=0}^{n} T_k$; so, by contradiction, suppose that $x \in \bigcup_{k=0}^{n} T_k$. Then $x = \sum_{i=1}^{r} a_i x'_i$ where $0 \leq x'_i < u_i$ for $i = 2, \ldots, r$ and $1 \leq \sum_{i=1}^{r} x'_i < 2^{n+1}$. By Lemma 4, there are integers α_i, $i = 1, 2, \ldots, r$ such that

$x_i = u_i\alpha_i + x'_i$ and $\sum_{i=1}^r \alpha_i = 0$. For $i = 2, \dots, r$, we have $\alpha_i \geq 0$ because on the contrary it would be $x_i = u_i\alpha_i + x'_i < 0$. Then, we have

$$2^{n+1} \leq \sum_{i=1}^r x_i = \sum_{i=1}^r (u_i\alpha_i + x'_i) = \sum_{i=1}^r u_i\alpha_i + \sum_{i=1}^r x'_i < 2^{n+1} + \sum_{i=1}^r u_i\alpha_i,$$

that is, $\sum_{i=1}^r u_i\alpha_i > 0$. Now, $\alpha_1 = -\sum_{i=2}^r \alpha_i$, so $\sum_{i=1}^r u_i\alpha_i = \sum_{i=2}^r (u_i - u_1)\alpha_i \leq 0$, which is a contradiction.

To show that $\partial^{n+1}S \subseteq S_{n+1}$, let $x \in \partial^{n+1}S = S \setminus \bigcup_{k=0}^n T_k$. By Lemma 4, x can be represented in the form $x = \sum_{i=1}^r a_i x_i$ where $0 \leq x_i < u_i$, $i = 2, \dots, r$. If $\sum_{i=1}^r x_i < 2^{n+1}$, then $x \in \bigcup_{k=0}^n T_k$, which is absurd. Therefore $\sum_{i=1}^r x_i \geq 2^{n+1}$, and this shows that $x \in S_{n+1}$. Thus, we have shown that

$$\partial^{n+1}S = \left\{ \sum_{i=1}^r a_i x_i : x_i \geq 0, \sum_{i=1}^r x_i \geq 2^n \right\} \cup \{0\}.$$

Now we prove that $\beta(S_{n+1}) = T_{n+1}$. First, if $x, y \in T_{n+1}$, then $x = \sum_{i=1}^r a_i x_i$ and $y = \sum_{i=1}^r a_i y_i$ where $0 \leq x_i < u_i, 0 \leq y_i < u_i$ for $i = 2, \dots, r$, $2^{n+1} \leq \sum_{i=1}^r x_i < 2^{n+2}$ and $2^{n+1} \leq \sum_{i=1}^r y_i < 2^{n+2}$. So, $x + y = \sum_{i=1}^r a_i(x_i + y_i)$ and $\sum_{i=1}^r (x_i + y_i) \geq 2^{n+2}$. If $x + y \in T_{n+1}$, then $x + y = \sum_{i=1}^r a_i x'_i$ where $0 \leq x'_i < u_i$ for $i = 2, \dots, r$ and $\sum_{i=1}^r x'_i < 2^{n+2}$. By Lemma 2, there are integers α_i, $i = 1, 2, \dots, r$ such that $x_i + y_i = u_i\alpha_i + x'_i$ and $\sum_{i=1}^r \alpha_i = 0$, where $\alpha_i \geq 0$ for $i = 2, \dots, r$, and this yields to

$$2^{n+2} \leq \sum_{i=1}^r (x_i + y_i) = \sum_{i=1}^r u_i\alpha_i + \sum_{i=1}^r x'_i < 2^{n+2} + \sum_{i=1}^r u_i\alpha_i,$$

that is, $\sum_{i=1}^r u_i\alpha_i > 0$. Now, since $\alpha_1 = -\sum_{i=2}^r \alpha_i$, we have $\sum_{i=1}^r u_i\alpha_i = \sum_{i=2}^r (u_i - u_1)\alpha_i \leq 0$, which is a contradiction. Thus, $x + y \notin T_{n+1}$. This shows that $\beta(\langle T_{n+1} \rangle) = T_{n+1}$.

It remains to show that T_{n+1} generates S_{n+1}. Since $T_{n+1} \subseteq S_{n+1}$, $\langle T_{n+1} \rangle \subseteq S_{n+1}$. To show that $S_{n+1} \subseteq \langle T_{n+1} \rangle$, let us take $x \in S_{n+1}$. By Lemma 4, x can be represented as $x = \sum_{i=1}^r a_i x_i$, where $0 \leq x_i < u_i$ for $i = 2, \dots, r$. It cannot occur that $\sum_{i=1}^r x_i < 2^{n+1}$ because $x \in S_{n+1}$. Thus, we have $\sum_{i=1}^r x_i \geq 2^{n+1}$.

Write $\sum_{i=1}^r x_i = q2^{n+1} + s$, where $0 \leq s < 2^{n+1}$ and $q \geq 1$. For $j = 1, 2, \dots, q + 1$ and $i = 1, 2, \dots, r$ we can find non-negative integers y_{ij} such that

$$\sum_{i=1}^r y_{ij} = 2^{n+1}, j = 1, \dots, q; \sum_{i=1}^r y_{i(q+1)} = s \text{ and } \sum_{j=1}^{q+1} y_{ij} = x_i, i = 1, 2, \dots, r.$$

Thus, we have $y_j := \sum_{i=1}^r a_i y_{ij} \in T_{n+1}$, for $j = 1, \dots, q - 1$ and $y_q := \sum_{i=1}^r a_i (y_{iq} + y_{i(q+1)}) \in T_{n+1}$. We see that $x = \sum_{j=1}^q y_j$, which shows that $x \in \langle T_{n+1} \rangle$. This ends the proof. □

For each $(r-1)$-tuple of the form $(x_2, x_3, \dots, x_r)$, where $0 \leq x_i < u_i$, $i = 2, 3, \dots, r$, let $s = x_2 + \cdots + x_r$ and let us define the following set

$$A(x_2, x_3, \dots, x_r) := \{(k-s)a_1 + x_2a_2 + \cdots + x_ra_r : 2^n \le k < 2^{n+1}\}.$$

If $s > k$, then $(k-s)a_1 + x_2a_2 + \cdots + x_ra_r \notin S$. In fact, if

$$(k-s)a_1 + x_2a_2 + \cdots + x_ra_r = y_1a_1 + y_2a_2 + \cdots + y_ra_r,$$

where $y_1 \ge 0$ and $0 \le y_i < u_i$, $i = 2, \dots, r$; then, by Lemma 3, it follows that $y_1 = k - s < 0$, a contradiction.

The sets $A(x_2, x_3, \dots, x_r)$ are pairwise disjoint (by Lemma 3) and each of them has 2^n elements. Therefore, $\bigcup A(x_2, x_3, \dots, x_r)$ (this union runs over all tuples of the form $(x_2, x_3, \dots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \dots, r$) has $2^n u_2 \cdots u_r = 2^n a_1$ elements. Note that $\beta(\partial^n S)$ is the set of elements in $\bigcup A(x_2, x_3, \dots, x_r)$ of the form $(k-s)a_1 + x_2a_2 + \cdots + x_ra_r$ for which $k \ge s$, where $s = x_2 + \cdots + x_r$.

For each tuple $(x_2, x_3, \dots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \dots, r$, and $n \ge 0$, we define $\alpha(x_2, x_3, \dots, x_r, n)$ to be the number of elements of the form $(k-s)a_1 + x_2a_2 + \cdots + x_ra_r$, where $s = x_2 + \cdots + x_r$, such that $2^n \le k < 2^{n+1}$ and $k \ge s$. Note that $0 \le \alpha(x_2, x_3, \dots, x_r, n) \le 2^n$. Thus,

$$e(\partial^n S) = \sum \alpha(x_2, x_3, \dots, x_r, n).$$

The last sum is taken over all tuples $(x_2, x_3, \dots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \dots, r$. For instance, when $r = 2$, we have

$$\alpha(x_2, n) = \begin{cases} 2^n, & \text{if } x_2 \le 2^n; \\ 2^{n+1} - x_2, & \text{if } 2^n < x_2 < 2^{n+1}; \\ 0, & \text{if } 2^{n+1} \le x_2, \end{cases}$$

and

$$\begin{aligned} e(\partial^n S) &= \sum_{x_2=0}^{u_2-1} \alpha(x_2, n) \\ &= \begin{cases} 2^n u_2, & \text{if } u_2 - 1 \le 2^n; \\ 2^{n+1}u_2 - 2^{2n-1} - 2^{n-1} - \frac{(u_2-1)u_2}{2}, & \text{if } 2^n < u_2 - 1 < 2^{n+1}; \\ 3 \cdot 2^{2n-1} + 2^{n-1}, & \text{if } 2^{n+1} \le u_2 - 1. \end{cases} \end{aligned}$$

Proposition 5 *Let $S = \langle a_1, a_2, \dots, a_r \rangle$. For $n \ge 0$, $\partial^n S$ has maximal embedding dimension if and only if $u_2 + \cdots + u_r \le 2^n + r - 1$.*

Proof The condition that $\partial^n S$ has maximal embedding dimension is equivalent to the equality $\alpha(x_2, x_3, \dots, x_r, n) = 2^n$ for all tuples $(x_2, x_3, \dots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \dots, r$. This means that $k \ge x_2 + x_3 + \cdots + x_r$ for all $2^n \le k < 2^{n+1}$ and all tuples $(x_2, x_3, \dots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \dots, r$. By taking $x_i = u_i - 1$, $i = 2, \dots, r$, and $k = 2^n$, we obtain $u_2 + \cdots + u_r - r + 1 \le 2^n$. It is clear that if

$u_2 + \cdots + u_r - r + 1 \le 2^n$, then $k \ge x_2 + x_3 + \cdots + x_r$ for all $2^n \le k < 2^{n+1}$ and all tuples $(x_2, x_3, \ldots, x_r)$, where $0 \le x_i < u_i$, $i = 2, \ldots, r$. This ends the proof. □

Proposition 6 *Let $S = \langle a_1, a_2, \ldots, a_r\rangle$. If $u_2 + \cdots + u_r \le 2^n + r - 1$, then $\partial^n S$ is β-symmetric.*

Proof Note that $\max \beta(\partial^n S) = (2^{n+1} - 1)a_1$ and

$$\min \beta(\partial^n S) = \left(2^n - \sum_{i=2}^{r}(u_i - 1)\right)a_1 + \sum_{i=2}^{r}(u_i - 1)a_i. \tag{5}$$

Note that the condition $u_2 + \cdots + u_r \le 2^n + r - 1$ implies that the coefficient of a_1 in the right hand side of (5) is a non-negative integer. Now, every element in $\beta(\partial^n S)$ has the form $(k - s)a_1 + a_2x_2 + \cdots + a_rx_r$, where $0 \le x_i < u_i$, $i = 2, \ldots, r$, $s = x_2 + \cdots + x_r$, $2^n \le k < 2^{n+1}$ and $k \ge s$. We must show that the element

$$t := \max \beta(\partial^n S) + \min \beta(\partial^n S) - [(k - s)a_1 + a_2x_2 + \cdots + a_rx_r]$$

belongs to $\beta(\partial^n S)$. In fact, we see that

$$\begin{aligned} t &= \left(2^{n+1} - 1 + 2^n - \sum_{i=2}^{r}(u_i - 1) - k + s\right)a_1 + \sum_{i=2}^{r}(u_i - 1 - x_i)a_i \\ &= \left(3 \cdot 2^n - 1 - \sum_{i=2}^{r}(u_i - 1) - k + s\right)a_1 + \sum_{i=2}^{r}(u_i - 1 - x_i)a_i. \end{aligned}$$

Let $s' = \sum_{i=2}^{r}(u_i - 1 - x_i)$ and $k' = 3 \cdot 2^n - 1 - k$. Observe that $0 \le u_i - 1 - x_i < u_i$ for $i = 2, \ldots, r$ and $2^n \le 3 \cdot 2^n - 1 - k < 2^{n+1}$, that is $2^n \le k' < 2^{n+1}$. Now,

$$k' - s' = (3 \cdot 2^n - 1 - k) - \sum_{i=2}^{r}(u_i - 1 - x_i) = 3 \cdot 2^n - 1 - \sum_{i=2}^{r}(u_i - 1) - k + s.$$

Therefore, $t = (k' - s')a_1 + \sum_{i=2}^{r}(u_i - 1 - x_i)a_i$. We also have $k' = 3 \cdot 2^n - 1 - k \ge \sum_{i=2}^{r}(u_i - 1 - x_i) = s'$, by hypothesis. This proves that $t \in \beta(\partial^n S)$. □

4 The conjecture $e(\partial S) \ge 2e(S) + 1$

Conjecture 2 says that for all numerical semigroup S without maximal embedding dimension, the inequality $e(\partial S) \ge 2e(S) + 1$ holds. Equality holds for some numerical semigroups. For instance, let $a, b > 1$ be relatively prime, $T = \langle a, b\rangle$ and assume that $F(T) = (a - 1)(b - 1) - 1 > a, b$. This implies that if $S = \langle a, b, F(T)\rangle = T \cup \{F(T)\}$, then $\beta(S) = \{a, b, F(T)\}$. Thus, $e(S) = 3$, $\partial S = \partial T$ and $e(\partial S) = e(\partial T) = 7$.

In this section we prove that Conjecture 2 is true if S is supersymmetric or $e(S) = 3$. In fact, in the case S is supersymmetric, we have the following result.

Theorem 3 *If S is a supersymmetric numerical semigroup that does not have maximal embedding dimension, then $e(\partial S) \geq 2e(S) + 3$. Equality holds if and only if $e(S) = 2$.*

Proof If $S = \langle a_1, a_2, \dots, a_r \rangle$ is supersymmetric as in Theorem 2, then $\beta(\partial S)$ is the set of all elements of the form $(k - s)a_1 + x_2a_2 + \cdots + x_ra_r$, where $0 \leq x_i < u_i$, $i = 2, \dots, r$, $s = x_2 + \cdots + x_r$, $k \in \{2, 3\}$ and $k \geq s$. By counting the number of elements in $\beta(\partial S)$, we find that $e(\partial S) = 1 - r + 2r^2 + \binom{r-1}{2}$. Now, if we assume that S has not maximal embedding dimension, then $r \geq 2$. Thus, we have

$$e(\partial S) = 1 - r + 2r^2 + \binom{r-1}{2} \geq 1 - r + 2r^2 \geq 2r + 3 = 2e(S) + 3$$

(in the second inequality we use that $r \geq 2$). It is easy to see that equality $e(\partial S) = 2e(S) + 3$ holds if and only if $e(S) = 2$. □

Before we start proving Conjecture 2 for the case of dimension 3, we prove the following lemma, that gives us a generating set of ∂S in general.

Lemma 5 *Let S be minimally generated by $a_1, a_2, \dots, a_r$, where $r = e(S)$. Then, the elements of the form $a_1x_1 + a_2x_2 + \cdots + a_rx_r$, where $x_1 + x_2 + \cdots + x_r \in \{2, 3\}$, generate ∂S.*

Proof Every nonzero element in ∂S can be represented as a sum $s_1 + s_2 + \cdots + s_t$, where $s_i \in \beta(S) = \{a_1, a_2, \dots, a_r\}$, $i = 1, 2, \dots, t$, with $t \geq 2$. We have two cases depending on the parity of t. If $t = 2q$ for some $q \geq 1$, then $s_1 + s_2 + \cdots + s_t = (s_1 + s_2) + (s_3 + s_4) + \cdots + (s_{t-1} + s_t)$. If $t = 2q + 3$ for some $q \geq 0$, then $s_1 + s_2 + \cdots + s_t = (s_1 + s_2) + (s_3 + s_4) + \cdots + (s_{t-4} + s_{t-3}) + (s_{t-2} + s_{t-1} + s_t)$. In any case, the element $s_1 + s_2 + \cdots + s_t$ can be represented a sum of elements of the form $a_1x_1 + a_2x_2 + \cdots + a_rx_r$, where $x_1 + x_2 + \cdots + x_r \in \{2, 3\}$. □

For the rest of this section, let $S = \langle a_1, a_2, a_3 \rangle$, with $\beta(S) = \{a_1, a_2, a_3\}$ and $3 < a_1 < a_2 < a_3$. Our purpose is to prove that ∂S has at least 7 minimal generators. By Lemma 5, the minimal generators of ∂S have the form $xa_1 + ya_2 + za_3$, where $x + y + z \in \{2, 3\}$. We set

$$G(S) = \{xa_1 + ya_2 + za_3 \, : \, x + y + z \in \{2, 3\}\}.$$

In general, $G(S)$ has at most 16 elements, which implies that $e(\partial S) \leq 16$. The equality is achieved, for instance with $S = \langle 14, 16, 19 \rangle$, where $\beta(\partial S) = \{28, 30, 32, 33, 35, 38, 42, 44, 46, 47, 48, 49, 51, 52, 54, 57\}$ has 16 elements.

Let $a \in G(S)$ and assume that $a \notin \beta(\partial S)$. Then a can be expressed as a linear combination of the elements in $G(S)\backslash\{a\}$. This implies that a can be written in the form

$$a = c_1 a_1 + c_2 a_2 + c_3 a_3, \tag{6}$$

where $c_i \geq 0$. We will call any representation of a as in (6) an *L- representation of a*. For instance, let us say that $a \in G(S)$ can be represented as $a = r(a_1 + a_2) + s(a_1 + a_3) + t(3a_2)$, then $a = (r+s)a_1 + (r+3t)a_2 + sa_3$ is an L-representation of a. For simplicity, when we have an L-representation of a, say $a = c_1a_1 + c_2a_2 + c_3a_3$, we will simply say that we have an L-representation $a = c_1a_1 + c_2a_2 + c_3a_3$.

Lemma 6 *Suppose that $\{i,j,k\} = \{1,2,3\}$. Let $a \in G(S)$. Assume that $a \notin \beta(\partial S)$ and that a can be expressed as a linear combination of a_i and a_j. Then, in every L-representation of a, the coefficient of a_k is positive.*

Proof The submonoid $\langle a_i, a_j\rangle$ of $\mathbb{N}$ is isomorphic to the numerical semigroup $\langle a_i/d, a_j/d\rangle$, where $d = \gcd(a_i, a_j)$. The numerical semigroup $\langle a_i/d, a_j/d\rangle$ is supersymmetric, so

$$\beta(\partial\langle a_i/d, a_j/d\rangle) = \{2a_i/d, 2a_j/d, a_i/d + a_j/d, 3a_i/d, 3a_j/d, 2a_i/d + a_j/d, a_i/d + 2a_j/d\}.$$

Therefore, the monoid $\langle a_i, a_j\rangle\backslash\{a_i, a_j\}$ is minimally generated by $\{2a_i, 2a_j, a_i + a_j, 3a_i, 3a_j, 2a_i + a_j, a_i + 2a_j\}$.

Now, if $a = c_1a_1 + c_2a_2 + c_3a_3$ is an L-representation of a and $c_k = 0$, then we have $a \in \{2a_i, 2a_j, a_i + a_j, 3a_i, 3a_j, 2a_i + a_j, a_i + 2a_j\}$ and a can be represented as a linear combination of elements in $\{2a_i, 2a_j, a_i + a_j, 3a_i, 3a_j, 2a_i + a_j, a_i + 2a_j\}\backslash\{a\}$, which is absurd. □

Lemma 7 $2a_1, a_1 + a_2 \in \beta(\partial S)$.

Proof Since $2a_1 = \min(\partial S\backslash\{0\})$, we have $2a_1 \in \beta(\partial S)$. Now, $a_1 + a_2 = \min(\partial S\backslash\{0, 2a_1\})$, so it is impossible to write $a_1 + a_2$ as a sum of nonzero elements of ∂S, unless $a_1 + a_2$ is a multiple of $2a_1$, which is not possible either. □

Note that the proof of Lemma 7 does not use the hypothesis that S has dimension 3. Therefore, in general, if S is minimally generated by $a_1 < a_2 < \cdots < a_r$, where $r \geq 2$, then $2a_1, a_1 + a_2 \in \beta(\partial S)$.

Lemma 8 $3a_1 \in \beta(\partial S)$.

Proof If $3a_1 \notin \beta(\partial S)$, then there is an L-representation $3a_1 = c_1a_1 + c_2a_2 + c_3a_3$. By Lemma 6, $c_2 > 0$ and $c_3 > 0$. It follows that $c_1 = 0, c_2 = 1$ and $c_3 = 1$, so $3a_1 = a_2 + a_3$. Taking an L-representation $3a_1 = a_2 + a_3 = d_1a_1 + d_2a_2 + d_3a_3$, by Lemma 6, $d_1 > 0, d_2 > 0, d_3 > 0$, which implies that $d_1a_1 + d_2a_2 + d_3a_3 \geq a_1 + a_2 + a_3 > 3a_1$. This is a contradiction. □

Lemma 9 $2a_1 + a_2 \in \beta(\partial S)$.

Proof If $2a_1 + a_2 \notin \beta(\partial S)$, there is an L-representation $2a_1 + a_2 = c_1a_1 + c_2a_2 + c_3a_3$. By Lemma 6, $c_3 > 0$. We note that $c_1 < 2$. If $c_1 = 1$, then $c_2 = 0$ (on the contrary, $c_1a_1 + c_2a_2 + c_3a_3 \geq a_1 + a_2 + a_3 > 2a_1 + a_2$). Thus, $2a_1 + a_2 = a_1 + c_3a_3$ and $a_1 + a_2 = c_3a_3$, which is impossible. Then, we have $c_1 = 0$. That is, $2a_1 + a_2 = c_2a_2 + c_3a_3$. If $c_2 \geq 2$, then $c_2a_2 + c_3a_3 \geq 2a_2 + a_3 > 2a_1 + a_2$, absurd. If $c_2 = 1$, then $c_3 = 1$; so, $2a_1 + a_2 = a_2 + a_3$, absurd. If $c_2 = 0$, then $2a_1 + a_2 = c_3a_3$ and $c_3 = 2$. Thus, we have $2a_1 + a_2 = 2a_3$. Now, there is an L-representation $2a_1 + a_2 = 2a_3 = d_1a_1 + d_2a_2 + d_3a_3$, where $d_1 > 0, d_2 > 0$ and $d_3 > 0$. So, we have $d_1a_1 + d_2a_2 + d_3a_3 \geq a_1 + a_2 + a_3 > 2a_1 + a_2$, which is absurd. □

Lemma 10 $2a_2 \in \beta(\partial S)$ *or* $a_1 + a_3 \in \beta(\partial S)$.

Proof Assume that $2a_2, a_1 + a_3 \notin \beta(\partial S)$. Then there is an L-representation $2a_2 = c_1a_1 + c_2a_2 + c_3a_3$, where $c_1 > 0$ and $c_3 > 0$. Besides, $c_3 < 2$, so $c_3 = 1$ and $2a_2 = c_1a_1 + c_2a_2 + a_3$. Note that c_2 cannot be positive, so that $2a_2 = c_1a_1 + a_3$. On the other hand, there is an L-representation $a_1 + a_3 = d_1a_1 + d_2a_2 + d_3a_3$, where $d_2 > 0$. It must be $d_1 = d_3 = 0$, so $a_1 + a_3 = d_2a_2$, where $d_2 \geq 2$. This is incompatible with $2a_2 = c_1a_1 + a_3$ unless $c_1 = 1$ and $d_2 = 2$. Thus, we have $2a_2 = a_1 + a_3$.

Finally, if $2a_2 = a_1 + a_3$ does not belong to $\beta(\partial S)$, then in all L-representations $2a_2 = a_1 + a_3 = e_1a_1 + e_2a_2 + e_3a_3$, we must have $e_1 > 0, e_2 > 0$ and $e_3 > 0$, but this is impossible. This ends the proof. □

By straightforward calculations, as in the last three lemmas, we obtain the following result.

Lemma 11 $a_1 + 2a_2 \in \beta(\partial S)$ *or* $2a_1 + a_3 \in \beta(\partial S)$.

By combining Lemmas 10 and 11, we obtain the following.

Proposition 7 *At least one of the following holds:*

1. $2a_2 \in \beta(\partial S), a_1 + a_3 \in \beta(\partial S)$ *and* $2a_2 \neq a_1 + a_3$.
2. $2a_2 \in \beta(\partial S)$ *and* $a_1 + 2a_2 \in \beta(\partial S)$.
3. $a_1 + a_3 \in \beta(\partial S)$ *and* $a_1 + 2a_2 \in \beta(\partial S)$.

Proof If $a_1 + 2a_2 \notin \beta(\partial S)$, then $2a_2 = 2a_1 + a_3 \in \beta(\partial S)$ by Lemma 11. If $a_1 + a_3 \notin \beta(\partial S)$, then, as in the proof of Lemma 10, $a_1 + a_3 = d_2a_2$ where $d_2 \geq 2$, so $2a_2 = a_1 + d_2a_2$, which is absurd. So, if $a_1 + 2a_2 \notin \beta(\partial S)$, then $2a_2 = 2a_1 + a_3, a_1 + a_3 \in \beta(\partial S)$ and $2a_2 \neq a_1 + a_3$. Finally, if $a_1 + 2a_2 \in \beta(\partial S)$, then Lemma 10 gives us the last two options. □

By Lemmas 7, 8 and 9, there are at least 4 minimal generators in ∂S, namely, $2a_1, a_1 + a_2, 3a_1$ and $a_1 + 2a_2$. Each case in Proposition 7 gives us two different minimal generators of ∂S, but it is possible that $3a_1 = 2a_2$.

Proposition 8 *If $3a_1 = 2a_2$, then ∂S has at least 7 minimal generators.*

Proof The condition $3a_1 = 2a_2$ implies that there exists $m > 1$ such that $a_1 = 2m$, $a_2 = 3m$ and $\gcd(a_3, m) = 1$. Now, we have already 4 minimal generators in ∂S: $2a_1 = 4m, a_1 + a_2 = 5m, 3a_1 = 6m, 2a_1 + a_2 = 7m$. Note that $3a_2 = 9m$ and $a_1 + 2a_2 = 8m$ are not minimal generators. The other possible minimal generators of ∂S have a_3 as a summand. They are $a_1 + a_3, a_2 + a_3, 2a_1 + a_3, 2a_3, 2a_2 + a_3, a_1 + 2a_3, a_2 + 2a_3, a_1 + a_2 + a_3$. By the division algorithm, $a_3 = sm + t$ where $0 \le t < m$ and $\gcd(m, t) = 1$. It must be $s \ge 3$. We note that $a_1 + a_3 < a_2 + a_3$ and the other 6 possible minimal generators are greater than $a_2 + a_3$. Now, $a_1 + a_3$ cannot be written as a linear combination of $2a_1, a_1 + a_2, 3a_1$ and $2a_1 + a_2$ because that would imply that a_3 is multiple of m. Thus, $a_1 + a_3 \in \beta(\partial S)$. Now, $a_2 + a_3$ cannot be a linear combination of $2a_1, a_1 + a_2, 3a_1$ and $2a_1 + a_2$ for similar reasons. So, if $a_2 + a_3 \notin \beta(\partial S)$, then $a_2 + a_3$ can be represented as a linear combination of $2a_1, a_1 + a_2, 3a_1, 2a_1 + a_2$ and $a_1 + a_3$, where the coefficient of $a_1 + a_3$ is positive; but this would imply that a_2 can be written as linear combination of a_1, a_2, a_3 with positive coefficient in a_1 and non-negative coefficients in a_2 and a_3, which is absurd. This shows that $a_2 + a_3 \in \beta(\partial S)$. Now, $2a_2 + a_3 = 3a_1 + a_3 = 2a_1 + (a_1 + a_3)$ does not belong to $\beta(\partial S)$. We prove now that $2a_1 + a_3$ belongs to $\beta(\partial S)$. If $2a_1 + a_3 \notin \beta(\partial S)$, then there is an L-representation $2a_1 + a_3 = c_1 a_1 + c_2 a_2 + c_3 a_3$, where $c_2 > 0$. If $c_3 > 0$, then $c_2 + c_3 \le 2$, which implies that $c_2 = c_3 = 1$. Thus, $2a_1 + a_3 = c_1 a_1 + a_2 + a_3$ and $2a_1 = c_1 a_1 + a_2$, but this is impossible. Therefore, $c_3 = 0$, which means that $2a_1 + a_3$ is representable as a linear combination of $2a_1, a_1 + a_2, 3a_1, 2a_1 + a_2$, but this implies that a_3 is a multiple of m, a contradiction. Thus, we have shown that there are at least 7 minimal generators in ∂S. □

By Proposition 8, if $3a_1 = 2a_2$, then ∂S has at least 7 minimal generators. Now, we can assume that $3a_1 \neq 2a_2$. In this case, by Proposition 7, we have at least 6 generators in ∂S. It only remains to show that there is at least one more minimal generator in ∂S. It is important to note that we have not used the condition of not having maximal embedding dimension yet. In order to finish, we need the following result.

Lemma 12 *If $a_1 = 4, a_2 = k + c$ and $a_3 = 3k - c$, where $k \ge 3$ and $c > 0$, then ∂S has at least 7 minimal generators.*

Proof Note that k and c must have different parities. We claim that $2a_2, a_1 + a_3, a_1 + 2a_2 \in \beta(\partial S)$. In fact, if $2a_2 \notin \beta(\partial S)$, then $2a_2 = d_1 a_1 + a_3$ for some $d_1 > 0$. That is, $2k + 2c = 4d_1 + 3k - c$, so $3c - k = 4d_1$. Therefore, k and c have the same parity, which is absurd.

Now, if $a_1+a_3 \notin \beta(\partial S)$, then $a_1+a_3=d_2a_2$ for some $d_2 \geq 2$; so, $4+3k-c=d_2k+d_2c$. Thus, $4+(3-d_2)k=(d_2+1)c \geq 3$, from which $(3-d_2)k \geq -1$. But, $(3-d_2)k=-1$ is impossible, so we must have $d_2 \leq 3$. Thus, we have two cases:

1. If $d_2=2$, then $4+3k-c=2k+2c$. So, $4+k=3c$, which implies that k and c have the same parity, absurd.
2. If $d_2=3$, then $a_1+a_3=3a_2$. This implies that $a_1+a_3 \in \beta(\partial S)$ (see the last paragraph of the proof of Lemma 10), which is absurd.

If $a_1+2a_2 \notin \beta(\partial S)$, then $a_1+2a_2=d_1a_1+d_2a_2+d_3a_3$ where $d_3>0$. Note that $d_2<2$. We have two cases:

1. If $d_2=0$, then $a_1+2a_2=d_1a_1+d_3a_3$. If $d_1>0$, then $2a_2=(d_1-1)a_1+d_3a_3$. It follows that $d_3=1$. That is, $2a_2=(d_1-1)a_1+a_3$. In terms of k and c, we obtain $3c=4(d_1-1)+k$. The last equation implies that k and c have the same parity, which is absurd. This shows that $d_1=0$. Therefore, $a_1+2a_2=d_3a_3$. It must be $d_3=2$, so $a_1+2a_2=2a_3$, which implies that $a_1+2a_2=2a_3 \in \beta(\partial S)$, a contradiction.
2. If $d_2=1$, then $a_1+2a_2=d_1a_1+a_2+d_3a_3$, $a_1+a_2=d_1a_1+d_3a_3$; it must be $d_1=0$. Then, $a_1+a_2=d_3a_3$, and this is impossible.

Finally, we have at least 7 minimal generators in ∂S, namely, $2a_1, a_1+a_2$, $3a_1$, $2a_1+a_2$, $2a_2$, a_1+a_3, a_1+2a_2, unless $2a_2=a_1+a_3$. This condition implies that $3c=4+k$, which implies that k and c have the same parity, a contradiction. □

Theorem 4 *If S is a numerical semigroup with $e(S)=3$ and S does not have maximal embedding dimension, then $e(\partial S) \geq 7$.*

Proof By Lemmas 7, 8 and 9 and Proposition 7 along with the condition $3a_1 \neq 2a_2$, we have at least 6 different minimal generators in ∂S. Our seventh candidate is $3a_2$.

If $3a_2 \notin \beta(\partial S)$, then there is an L-representation $3a_2=c_1a_1+c_2a_2+c_3a_3$, where $c_1>0$ and $c_3>0$. It must be $c_2<2$. We have the following cases.

1. If $c_2=1$, then $3a_2=c_1a_1+a_2+c_3a_3$. It follows that $c_3=1$, that is, $3a_2=c_1a_1+a_2+a_3$. Now, suppose that $a_1+a_2+a_3 \notin \beta(\partial S)$. Then, there is an L-representation $a_1+a_2+a_3=d_1a_1+d_2a_2+d_3a_3$, and we deduce that two of the d_i's are zero and the other one is positive. This gives rise to three cases depending on which d_i is positive; but it is easy reach a contradiction in any case. For instance, in the case $d_1>0$ we have $a_1+a_2+a_3=d_1a_1$. It follows that $3a_2=(c_1+d_1-1)a_1$. As a_1 and a_2 are relatively prime (for the relation $3a_2=c_1a_1+a_2+a_3$), it must be $a_1=3$, a contradiction because $e(S)=3$ and S has not maximal embedding dimension. This proves that $3a_2 \in \beta(\partial S)$ or $a_1+a_2+a_3 \in \beta(\partial S)$.

   Now, in the three cases of Proposition 7 we have 6 different minimal generators for ∂S. Those cases combined with the two cases depending whether

$3a_2 \in \beta(\partial S)$ or $a_1 + a_2 + a_3 \in \beta(\partial S)$ give rise to 6 cases. In any of these 6 cases we obtain 7 minimal generators for ∂S, unless $3a_2 = a_1 + a_3$. Now, we have to prove that in case $3a_2 = a_1 + a_3$, we can find at least 7 minimal generators in ∂S. In fact, consider $a_2 + a_3$. If we show that $a_2 + a_3 \in \beta(\partial S)$, we are done. Suppose $a_2 + a_3 \notin \beta(\partial S)$. Then, there is an L-representation $a_2 + a_3 = d_1a_1 + d_2a_2 + d_3a_3$, where $d_1 > 0$. We see that it must be $d_2 = d_3 = 0$; so $a_2 + a_3 = d_1a_1$, and it is clear that $d_1 \geq 3$. By using that $3a_2 = a_1 + a_3$, we obtain $4a_2 = (d_1 + 1)a_1$. Now, the relation $3a_2 = a_1 + a_3$ implies that a_1 and a_2 are relatively prime. Since $a_1 > 3$, we have $a_1 = 4$, $a_2 = d_1 + 1$ and $a_3 = 3a_2 - a_1 = 3d_1 - 1$. Thus, by Lemma 12, ∂S has at least 7 minimal generators.

2. If $c_2 = 0$, then $3a_2 = c_1a_1 + c_3a_3$. Note that it must be $c_3 < 3$, so $1 \leq c_3 < 3$.

   Now, if $a_2 + a_3 \notin \beta(\partial S)$, then $a_2 + a_3 = ka_1$ for some $k \geq 3$. Then $3ka_1 = 3a_2 + 3a_3 = c_1a_1 + (c_3 + 3)a_3$, which reduces to $(3k - c_1)a_1 = (c_3 + 3)a_3$. Since a_1 and a_3 are relatively prime, a_1 divides $c_3 + 3$. But, $4 \leq c_3 + 3 < 6$ and $a_1 > 3$, so $a_1 = c_3 + 3$. Thus, $a_3 = 3k - c_1$. We have two cases.

(a) If $c_3 = 1$, then $a_1 = 4$, $a_3 = 3k - c_1$ and $a_2 = ka_1 - a_3 = k + c_1$. By Lemma 12, ∂S has at least 7 minimal generators.

(b) If $c_3 = 2$, then $3a_2 = c_1a_1 + 2a_3$. If $a_1 + 2a_3 \notin \beta(\partial S)$, then $a_1 + 2a_3 = d_1a_1 + d_2a_2 + d_3a_3$, where $d_2 > 0$. It must be $d_2 + d_3 \leq 2$. This gives us three cases; but it is easy to see that in each case we reach a contradiction. This shows that $3a_2 \in \beta(\partial S)$ or $a_1 + 2a_3 \in \beta(\partial S)$. These two cases combined with the three cases of Proposition 7 give rise to 6 cases. In all these cases, we obtain at least 7 minimal generators for ∂S, unless $3a_2 = a_1 + a_3$. But, we showed that under this condition, ∂S has at least 7 minimal generators.

□

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