International Journal of Electrical and Computer Engineering (IJECE)
Vol. 12, No. 1, February 2022, pp. 41∼49
ISSN: 2088-8708, DOI: 10.11591/ijece.v12i1.pp41-49

Convergence analysis of the triangular-based power flow method for AC distribution grids

María Camila Herrera[1], Oscar Danilo Montoya[2,3], Alexander Molina-Cabrera[4], Luis Fernando Grisales-Noreña[5], Diego Armando Giral-Ramírez[6]
[1]Ingeniería Eléctrica, Universidad Distrital Francisco José de Caldas, Bogotá D.C., Colombia
[2]Facultad de Ingeniería, Universidad Distrital Francisco José de Caldas, Bogotá D.C., Colombia
[3]Laboratorio Inteligente de Energía, Universidad Tecnológica de Bolívar, Cartagena, Colombia
[4]Facultad de Ingeniería, Universidad Tecnológica de Pereira, Pereira, Pereira, Colombia
[5]Departamento de Electromecánica y Mecratrónica, Instituto Tecnológico Metropolitano, Medellín, Colombia
[6]Facultad Tecnológica, Universidad Distrital Francisco José de Caldas, Bogotá D.C., Colombia

Article Info

Article history:

Received Feb 25, 2021
Revised Jun 3, 2021
Accepted Jun 30, 2021

Keywords:

Banach fixed-point theorem
Convergence analysis
Electric distribution networks
Triangular-based power flow method

ABSTRACT

This paper addresses the convergence analysis of the triangular-based power flow (PF) method in alternating current radial distribution networks. The PF formulation is made via upper-triangular matrices, which enables finding a general iterative PF formula that does not require admittance matrix calculations. The convergence analysis of this iterative formula is carried out by applying the Banach fixed-point theorem (BFPT), which allows demonstrating that under an adequate voltage profile the triangular-based PF always converges. Numerical validations are made, on the well-known 33 and 69 distribution networks test systems. Gauss-seidel, newton-raphson, and backward/forward PF methods are considered for the sake of comparison. All the simulations are carried out in MATLAB software.

Corresponding Author:

Oscar Danilo Montoya
Facultad de Ingeniería, Proyecto Curricular de Ingeniería Eléctrica, Universidad Distrital Francisco José de Caldas
Cra 7 # 40B-53, Bogotá D.C., Colombia
Email: odmontoyag@udistrital.edu.co

1. INTRODUCTION

Electrical distribution networks correspond to the part of a power system entrusted with providing electrical service to all the end-users in electric distribution grids operated at medium- and low-voltage levels [1], [2]. These grids are classically connected to substations that represent the interfaces with the power system at transmission and sub-transmission levels [3], [4]. The main characteristic of a distribution network is its typical radial topology, as this configuration facilitates minimizing investment costs [5]; furthermore, the radial design simplifies the creation of protection schemes, since the current typically flows from the substation to the loads [6], [7].

In the analysis of electrical systems, the concept of power flow (PF) emerges as the main tool to know the network's operative state under steady-state conditions [8], [9], i.e., voltage magnitudes and angles in the nodes and current magnitudes and angles in lines [10], [11]. The PF corresponds to a set of nonlinear equality restrictions that due to their complexity require iterative methods to find the numerical solution [12], [13]. In the specialized literature, the problem of PF has been addressed with multiple methodologies that include classical and accelerated gauss-seidel (GS) approaches and newton-raphson (NR) formulations [14],

[15], and convex formulations via semi-definite programming and second-order cone programming [16], [17]. However, the radial topology of the network and the presence of only one voltage-controlled source offer a particular structure which is taken into account for multiple PF in the literature; some of them are: successive approximations [13], backward/forward [9], graph-based methods with incidence matrices [3], and triangular-based approaches [12]. The main characteristic of these approaches is that they work directly with the nonlinear structure of the problem by proposing a recursive formula that allows calculating the voltages in all the demand nodes as a function of the power consumption [18], [19].

For some of these methods the convergence test is available in the literature, thereby ensuring that the method always converges to the PF solution under well-defined demand conditions. Some of these methods are GS [15], NR [14], successive approximations [13], and backward/forward [9] PF methods. For the triangular-based PF method, however, the convergence demonstration has not yet been performed, which is a gap this work tries to fill. The contributions of this work are: i) the analysis of convergence with the Banach fixed-point theorem (BFPT) in single-phase AC distribution networks; and ii) an analytical comparison with classical PF methods, which will provide evidence that the triangular-based PF method has lower processing times in numerically solving the PF problem.

It is worth mentioning that the studied triangular-based PF problem for AC grids has been initially proposed in [12] and [20] for three-phase unbalanced networks. However, a literature review provided no evidence of any convergence test in single-phase equivalent distribution grids. For this reason, we focus on such a convergence test by using the BFPT.

The rest of this document is structured as follows: section 2 presents the mathematical formulation of the triangular-based PF problem for electrical distribution systems by using a small test feeder working as an application example. Section 3 shows the convergence test via BFPT by exploiting the characteristics related with the diagonally dominant properties of the impedance matrix that relates all the nodes of the system. Section 4 describes the main features of the electrical distribution test feeders. Section 5 presents the computational validations, including the convergence test evaluation and the comparison with classical PF methodologies reported in the literature. Finally, section 6 offers the concluding remarks derived from this work and some possible future lines of investigation.

2. TRIANGULAR-BASED PF FORMULATION

This method is based on an upper-triangular matrix, which is set up using the relation between current flow through the lines and the current demand on the network nodes. The matrix is denoted using the letter T, and the general description is in [12], [21]. To clarify the basis of the method formulation and the use of the T matrix, we use a small electrical system constituted of 7 nodes and 6 lines and an operative voltage of 23 kV in the slack node, which is located at node 1. The configuration of this example is depicted in Figure 1. In addition, the parametric information of this test feeder is reported in Table 1.

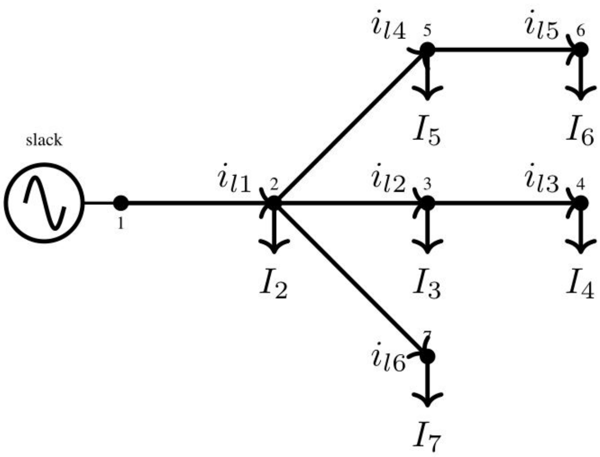

Figure 1. Schematic connection between nodes in 7-node test feeder used in the Triangular-based PF formulation example

Table 1. Electrical parameters of the 7-node test feeder used in the triangular-based PF formulation example

N_i	N_j	R_{ij} (Ω)	X_{ij} (Ω)	P_j (kW)	Q_j (kVAr)
1	2	0.5025	0.3025	1000	600
2	3	0.4020	0.2510	900	500
3	4	0.3660	0.1864	2500	1200
2	5	0.3840	0.1965	1200	950
5	6	0.8190	0.7050	1050	780
2	7	0.2872	0.4088	2000	1150

Once the values are organized as in the Table 1, the procedure for the the upper-triangular PF formulation is as follows: First, the T matrix is created for the network; to do so it is necessary to establish the relation among the currents through branches and the currents injected to the nodes. The relation between branch current i_b and the load consumption of each node I_n without taking into account the current of the slack node must be determined. For the 7-node test feeder, the flow direction of the currents was taken as shown in Figure 1, obtaining the set of (1).

$$\begin{aligned} i_{l1} &= I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \\ i_{l2} &= I_3 + I_4 \\ i_{l3} &= I_4 \\ i_{l4} &= I_5 + I_6 \\ i_{l5} &= I_6 \\ i_{l6} &= I_7 \end{aligned} \tag{1}$$

The second step is to change the set of equations as shown in (1) to matrix form as in (2); the matrix tying up the currents of the branches (i_b) and the currents of the nodes (I_n) is the upper-triangular T matrix. Then we can simplify (2) to a single equation as defined in (3).

$$\begin{bmatrix} i_{l1} \\ i_{l2} \\ i_{l3} \\ i_{l4} \\ i_{l5} \\ i_{l6} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} \tag{2}$$

$$i_b = T I_n \tag{3}$$

Similarly, we can find a relation between voltages of the nodes and branches by writing down the branch voltages in between any node and the slack node. For the proposed example, the set of (4) show the related voltages of the network, where V_0 is the voltage of the slack node.

$$\begin{aligned} V_2 &= V_0 - v_{l1} \\ V_3 &= V_0 - v_{l1} - v_{l2} \\ V_4 &= V_0 - v_{l1} - v_{l2} - v_{l3} \\ V_5 &= V_0 - v_{l1} - v_{l4} \\ V_6 &= V_0 - v_{l1} - v_{l4} - v_{l5} \\ V_7 &= V_0 - v_{l1} - v_{l6} \end{aligned} \tag{4}$$

Now, we should rewrite these equations in matrix form as shown in (5); once again the matrix tying up the voltages of the branches v_b and voltages of the nodes V_n is T, but this time in its transposed form. These relations lead us to (6), which is the equation for the voltages of the network.

$$\begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} V_0 - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{l1} \\ v_{l2} \\ v_{l3} \\ v_{l4} \\ v_{l5} \\ v_{l6} \end{bmatrix} \tag{5}$$

$$V_n = 1V_0 - T^T v_b \tag{6}$$

It is also necessary to find an equation describing the voltages of the branches; one way to do so is by multiplying the impedance of each branch with the current flowing through them, Z_b is a diagonal matrix formed with branch impedances of the network. The impedance matrix Z_b for the 7-node test feeder example is shown in (7).

$$\begin{bmatrix} 0.5025 + j0.3025 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0.2872 + j0.4088 \end{bmatrix} \tag{7}$$

In (8) describes the relation between Z_b and the voltages of the branches (v_b).

$$v_b = Z_b i_b \tag{8}$$

The next step is to use (3) in (8) to obtain (9). Then, replacing (9) in (6) we get the equation for the triangular-based PF method, as displayed in (10).

$$v_b = Z_b T I_n \tag{9}$$

$$V_n = 1V_0 - T^T Z_b T I_n \tag{10}$$

The main idea of this kind of load flow formulation is to find the voltage of all nodes in the network with information provided by the system itself. If we take a closer look at (10), all the variables required to calculate V_n can be defined by the details in Table 1, but I_n has to be rewritten in known terms. In (11) represents the net power demanded for any node. This one is listed in the information provided for the network; here, we need to rewrite it in matrix form so we can use it. In (12) shows a way to do so [15].

$$S_n = V_n I_n^* \tag{11}$$

$$I_n = \left(diag(V_n^*)\right)^{-1} S_n^* \tag{12}$$

In (12) the term $\left(diag(V_n^*)\right)^{-1}$ is the inverse of a diagonal matrix with conjugated nodal voltages, and as mentioned earlier S_n^* is the power demand for each node in its conjugated form.

The final equation to solve load flow based on an upper-triangular matrix is defined in (13); this is a nonlinear equation that requires an iterative process to be solved.

$$V_n = 1V_0 - T^T Z_b T \left(diag(V_n^*)\right)^{-1} S_n^* \tag{13}$$

The solution of expression (13) is found by adding an iterative counter t to the nodal voltages as (14):

$$V_n = 1V_0 - Z_n \left(diag(V_n^*)\right)^{-1} S_n^* \tag{14}$$

where, $Z_n = T^T Z_b T$, and the iterative process is carried out until the desired convergence is reached, i.e., $\left|\left|V_n^{t+1}\right| - \left|V_n^t\right|\right| \leq \epsilon$, where ϵ is the tolerance error, which is typically defined in the PF literature as 1×10^{-10} [20].

3. CONVERGENCE ANALYSIS

The convergence test presented in this section is based on the demonstration of the convergence for the successive approximations presented in [3]. For applying it on the iterative upper-triangular PF defined in (13), let us take into account the assumptions presented [22]:

Assumption 1 : The total connected load of the distribution grid does not produce a voltage collapse, i.e., the

power flow equations have a solution.
Assumption 2 : There is a positive lower voltage limit for all the voltages in the grid, $V^{\min} > 0$, which is assigned by the regulatory policies and utility practices.
Assumption 3 : The component of the impedance matrix that relates the demands among them, i.e., $\mathbb{Z}_n$, is diagonally dominant, which means that $\left|\mathbb{Z}_{n_{jj}}\right| \geq \left|\mathbb{Z}_{n_{jk}}\right|$, $\forall j \neq k$ is always guaranteed.

To demonstrate the properties of convergence of the triangular-based PF approach defined by (13), let us show the general definition of the BFPT as presented below [9], [22].
Lemma 1 [Banach fixed-point theorem]: The iterative formula defined by (13) is stable and it is a contraction map that takes the following structure.

$$V_n^{t+1} = g\left(V_n^t\right), \tag{15}$$

For some V that complies with Assumption 3. independent of the initial point, i.e., V^0, such that;

$$\left|\left|g\left(V_n^0\right) - g\left(U\right)\right|\right| \leq \gamma \left|\left|V^0 - U\right|\right| \tag{16}$$

where U represents the solution of the PF problem, and γ is a real number in the interval $[0, 1]$.
Proof: The recursive formula for the triangular-based PF method presented in (13) can be rewritten as presented in (17):

$$V_n^{t+1} = g\left(V_n^t\right) = 1V_0 - Z_n \left[\frac{S_{n_i}^\star}{V_{n_i}^{t,\star}}\right]_{i\in\Omega_S}^T \tag{17}$$

where Ω_S corresponds to the set that contains all the demand buses.

Additionally, considering the structure of the BFPT, it is possible to conclude that the solution of the PF problem, i.e., U, complies with $U = g\left(U\right)$, and this corresponds to a unique solution if and only if $g\left(U\right)$ represents a contraction mapping on V_n,

$$\begin{aligned} \left|\left|V_n^{t+1} - U\right|\right| &= \left|\left|g\left(V_n^{t+1}\right) - g\left(U\right)\right|\right| \\ &\left|\left|Z_n S_n^\star \left[\frac{1}{U_i^\star} - \frac{1}{V_{n_i}^{t,\star}}\right]_{i\in\Omega_S}^T\right|\right| \\ &\left|\left|Z_n S_n^\star\right|\right| \left|\left|\left[\frac{V_{n_i}^{t,\star} - U_i^\star}{U_i^\star V_{n_i}^{t,\star}}\right]_{i\in\Omega_S}^T\right|\right| \\ &\leq \gamma \left|\left|V_n^t - U\right|\right|, \end{aligned} \tag{18}$$

with

$$\gamma = \frac{\left|\left|Z_n S_n^\star\right|\right|}{\left(V^{\min}\right)^2}. \tag{19}$$

Now, taking into account assumption 3, which is associated with the nature of the impedance matrix, in (19) can be rewritten as (20);

$$\gamma = \max_{i\in\Omega_S} \left\{ \frac{\left|Z_{n_{ii}}\right| \left|S_{n_i}^\star\right|}{\left(V^{\min}\right)^2} \right\}. \tag{20}$$

It is worth mentioning that, taking into account the mathematical structure defined in (20) and that $Z_{n_{ii}}$ represents the equivalent impedance at node i (i.e., Thévenin equivalent), the next relation is attained.

$$\gamma = \max_{i\in\Omega_S} \left\{ \frac{\frac{\left|\mathbb{S}_{n_i}^\star\right|}{V^{\min}}}{\frac{V^{\min}}{\left|\mathbb{Z}_{n_{ii}}\right|}} \right\}, \tag{21}$$

Where we can guarantee that $0 \leq \gamma \leq 1$, as the denominator in (21) is the lower short-circuit current and the numerator represents the maximum consumption current, which in all the cases is lower than any short-circuit current that can appear during normal operating conditions. This demonstrates that the recursive triangular-based PF (13) solves the PF problem, which completes the proof [9], [20].

4. TEST SYSTEMS

This section presents the electrical configuration, as well as the test system information, of the radial distribution systems employed in this work for validating the triangular-based PF method. Two test systems are employed: the 33-node test system and the 69-node test feeder, which have radial structure. The complete information on these test systems is presented below. Note that for both test feeders we consider 12.66 kV and 1 MW of voltage and power bases, respectively.

4.1. 33-node test feeder

This electrical test feeder is constituted of 32 lines and 33 nodes with radial structure, and is operated with 12.66 kV at the substation bus. The schematic connection among nodes in this distribution grid is presented in Figure 2. The total demand of this system is 3715 kW and 2300 kVAr, which produces 210.9876 kW of power losses. For data of this test feeder regarding loads and branch parameters please consult [23].

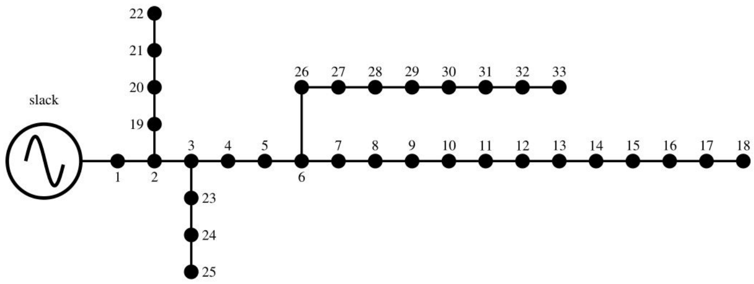

Figure 2. Schematic representation of the connection between nodes in the 33-node test feeder

4.2. 69-node test feeder

This electrical test feeder is composed of 68 lines and 69 nodes with radial structure, and is operated with 12.66 kV at the substation bus. The schematic connection among nodes in this distribution grid is presented in Figure 3. The total demand of this system is 3890.7 kW and 2693.6 kVAr, which produces 225.0718 kW of power losses. For data of this test feeder regarding loads and branch parameters please consult [23].

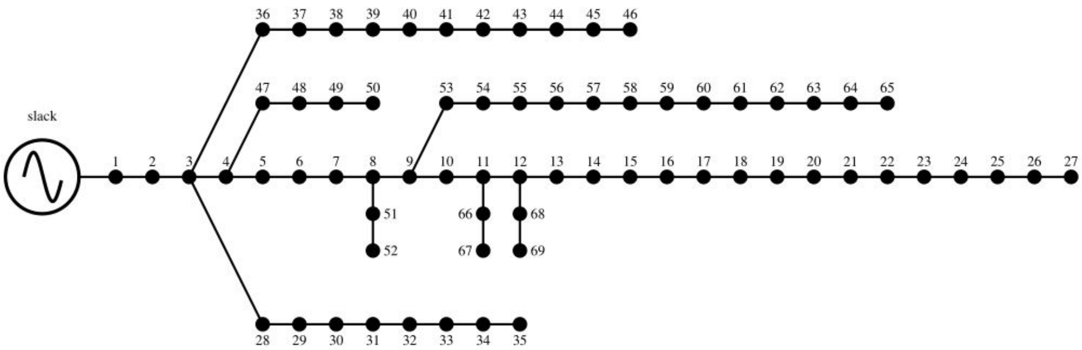

Figure 3. Schematic interconnection among nodes in the 69-node test system

5. COMPUTATIONAL VALIDATION

To validate the studied triangular-based PF for radial distribution networks with guarantee of convergence under well defined voltage conditions, we compare it with six classical methodologies reported in the literature: i) classical GS [14], ii) accelerated version of GS (AGS) [15], iii) NR [24], iv) Levenberg-marquardt (LM) [25], v) successive approximations (SA) [13], and vi) matricial backward-forward (MBF) [9], [3]. To perform a fair comparison, all these methods were run 1000 times to determine their computational effort (processing times) by considering a maximum of 100 iterations and a tolerance error of 1×10^{-10}. This process was done with MATLAB 2020a on a laptop with Intel Core$^{\text{TM}}$ i5-8300 2.3 Ghz and 4 GB RAM running a 64-bit.

In Table 2 is reported the numerical performance of the studied triangular-based PF and the other compared methods. From the results reported in this table we can observe that: i) The GS and its accelerated versions take higher computational times and number of iterations. This situation is attributable to the fact that these methods do not use matricial representations and they work with single equations, which implies that more calculations are needed at each iteration; ii) The NR and the Levenberg-Marquardt methods use a lower number of iterations to reach the PF solution. This is because these methods use Jacobian matrices in their formulations, which reduce the number of iterations in the searching process since information regarding derivatives is included in them. However, the inversion requirements of the Jacobian matrices at each iteration increase their processing times considerably when the size of the distribution network starts to increase; iii) The successive approximation, matricial backward/forward, and the triangular-based PF methods have the best computation performance in both test feeders, with the triangular method being the fastest one. In addition, the three methodologies are stabilized at the same number of iterations (10 iterations for both test feeders). This behavior is caused by the fact that these three methods work only with a reformulation of the power balance equations in complex form, and they do not include derivatives in their formulations; and iv) The main advantage of using the studied triangular-based PF formulation for PF calculations in AC radial distribution networks is that this method is faster compared with its alternatives. This is attributable to the fact that this PF formulation (13) does not intervene in any inverse of the admittance matrix, which reduces the computational effort during the iterative procedure. Figure 4 presents the voltage behaviors in both test feeders for the studied TB and MBF PF methods.

Table 2. Numerical results in the 33- and 69-node test feeders for the studied and comparative methods

	Method	Proc. Time [ms]	Iterations	Losses [kW]
33-node test feeder	GS	1148.422	2313	210.987
	AGS ($\alpha = 1.82$)	768.921	227	210.987
	NR	3.006	5	210.987
	LM	4.251	5	210.987
	SA	0.931	10	210.987
	MBF	0.936	10	210.987
	TB	0.717	10	210.987
69-node test feeder	GS	8430.077	32687	225.071
	AGS ($\alpha = 1.92$)	480.186	1628	225.071
	NR	14.304	5	225.071
	LM	14.704	5	225.071
	SA	3.329	10	225.071
	MBF	4.252	10	225.071
	TB	2.522	10	225.071

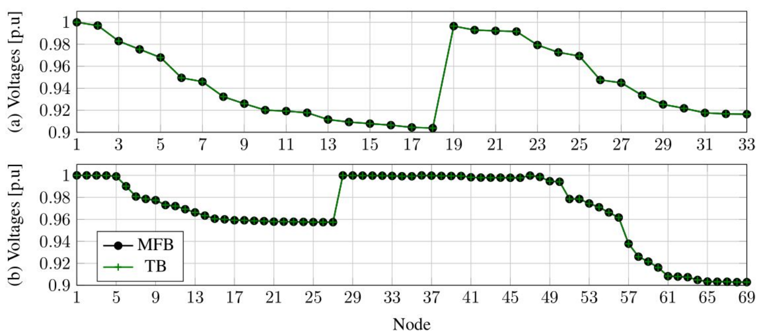

Figure 4. Voltage profiles; (a) 33-bus electrical network, and (b) 33-bus electrical network

Note that in all the compared methods and in the studied triangular-based PF formulation, the power solution is identical, since the estimated power losses are the same for all the methods (see the last column in Table 2). However, the selection of each of these methods depends on the analytical requirements, it being always more attractive to use the speediest one, since it can be embedded in real-time applications. We can note from Figure 4 that these curves confirm that, in a numerical sense, both PF methods present the same performance regarding voltage calculations. However, as shown in Table 2, the main advantage of the TB PF approach when compared to the MBF method is its quicker calculation in radial distribution networks, added to the fact that this method guarantees convergence on well-defined voltage conditions as demonstrated in section 3.

Finally, to show that the studied TB power flow method ensures convergence in both electrical distribution test feeders, Figure 5 presents the demeanor of the $\gamma-$coefficient. From Figure 5, we confirm that hypothesis (21) is guaranteed, and the studied TB power ensures the numerical solution of the PF problem in AC networks. For the 33-node test feeder, the maximum $\gamma-$coefficient is presented at node 30 with a numerical value of 0.027, while in the case of the 69-node test feeder, this value happens at node 61 with a numerical value of 0.062. These numbers make easy the verification that γ is contained in the interval $[0, 1]$ for both test feeders.

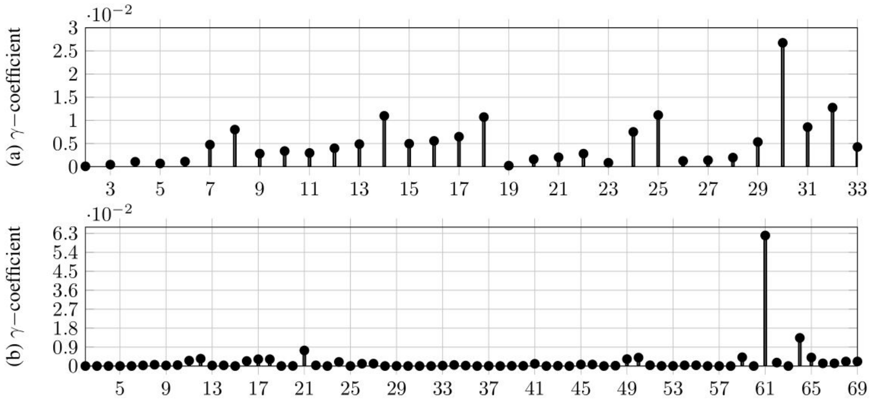

Figure 5. $\gamma-$coefficient performance; (a) 33-bus electrical network, and (b) 69-bus electrical network

6. CONCLUSION AND FUTURE WORK

This paper has addressed the convergence analysis of a PF method named as a triangular-based PF method for radial AC distribution networks. This PF formulation uses an upper-triangular matrix that relates branch and nodal currents to obtain a recursive formula that relates nodal voltages with constant power consumption, which does not require any matrix inversion, thus reducing the computational time in comparison with classical PF formulations. The BFPT was employed to demonstrate the convergence of this PF method based on the diagonally dominant properties of the Z_n impedance matrix. Numerical results showed that for both test feeders the γ-coefficient was always lower than unity, which helped confirm that the triangular-based PF is itself a contraction mapping. A possible future work will be the usage of the triangular-based PF formulation embedded in optimization procedures to address different distribution problems such as the optimal location of shunt devices (i.e., capacitors, distributed generators, and filters) to improve the solution times of these problems.

ACKNOWLEDGEMENT

This work was supported in part by the Centro de Investigación y Desarrollo Científico de la Universidad Distrital Francisco José de Caldas under grant 1643-12-2020 associated with the project: *"Desarrollo de una metodología de optimización para la gestión óptima de recursos energéticos distribuidos en redes de distribución de energía eléctrica,"* and in part by the Dirección de Investigaciones de la Universidad Tecnológica de Bolívar under grant PS2020002 associated with the project: *"Ubicación óptima de bancos de capacitores de paso fijo en redes eléctricas de distribución para reducción de costos y pérdidas de energía: Aplicación de métodos exactos y metaheurísticos."*

REFERENCES

[1] R. Poudineh, D. Peng, and S. Mirnezami, "Electricity networks," Oxford Institute for Energy Studies, techreport, Dec. 2017.

[2] M. S. Rawat and S. Vadhera, "Probabilistic approach to determine penetration of Hybrid Renewable DGs in Distribution Network Based on Voltage Stability Index," *Arabian Journal for Science and Engineering*, vol. 45, no. 3, pp. 1473–1498, jul 2019, doi: 10.1007/s13369-019-04023-1.

[3] O. D. Montoya, W. Gil-Gonzalez, and D. A. Giral, "On the matricial formulation of iterative sweep power flow for radial and meshed distribution networks with guarantee of convergence," *Applied Sciences*, vol. 10, no. 17, Aug. 2020, Art. No. 5802, doi: 10.3390/app10175802.

[4] J. A. D. Massignan, B. R. Pereira, and J. B. A. London, "Load flow calculation with voltage regulators Bidirectional Mode and Distributed Generation," *IEEE Transactions on Power Systems,* vol. 32, no. 1, pp. 1576-1577, 2016, doi: 10.1109/TPWRS.2016.2576679.

[5] K. Balamurugan and D. Srinivasan, "Review of power flow studies on distribution network with distributed generation," *IEEE Ninth Int. Conf. on Power Electro. and Drive Sys.,* Dec. 2011, doi: 10.1109/PEDS.2011.614728.

[6] R. Rajaram, K. S. Kumar, and N. Rajasekar, "Power system reconfiguration in a radial distribution network for reducing losses and to improve voltage profile using modified plant growth simulation algorithm with distributed generation (DG)," *Energy Reports*, vol. 1, pp. 116–122, Nov. 2015, doi: 10.1016/j.egyr.2015.03.002.

[7] M. W. Saddique, S. S. Haroon, S. Amin, A. R. Bhatti, I. A. Sajjad, and R. Liaqat, "Optimal placement and sizing of shunt capacitors in radial distribution system using polar bear optimization algorithm," *Arabian Journal for Science and Engineering*, vol. 46, pp. 873-899, Jul. 2020, doi: 10.1007/s13369-020-04747-5.

[8] M. E. Nassar and M. Salama, "A novel branch-based power flow algorithm for islanded AC microgrids," *Electric Power Systems Research*, vol. 146, pp. 51–62, May. 2017, doi: 10.1016/j.epsr.2017.01.019.

[9] T. Shen, Y. Li, and J. Xiang, "A graph-based power flow method for balanced distribution systems," *Energies*, vol. 11, no. 3, Feb. 2018, Art. No. 511, doi: 10.3390/en11030511.

[10] M. C. Herrera-Bri˜nez, O. D. Montoya, L. Alvarado-Barrios, and H. R. Chamorro, "The equivalence between successive approximations and matricial load flow formulations," *Applied Sciences*, vol. 11, no. 7, Mar. 2021, Art. No. 2905, doi: 10.3390/app11072905.

[11] S. H. Low, "Convex relaxation of optimal power flow-Part II: Exactness," *IEEE Trans. Control Netw. Syst*, vol. 1, no. 2, pp. 177–189, Jun. 2014, doi: 10.1109/TCNS.2014.2323634.

[12] A. Marini, S. Mortazavi, L. Piegari, and M.-S. Ghazizadeh, "An efficient graph-based power flow algorithm for electrical distribution systems with a comprehensive modeling of distributed generations," *Electric Power Systems Research*, vol. 170, pp. 229–243, May. 2019, doi: 10.1016/j.epsr.2018.12.026.

[13] O. D. Montoya and W. Gil-Gonzalez, "On the numerical analysis based on successive approximations for power flow problems in AC distribution systems," *Electric Power Systems Research*, vol. 187, Oct. 2020, Art. No. 106454, doi: 10.1016/j.epsr.2020.106454.

[14] J. J. Grainger and W. D. Stevenson, "Power system analysis," ser. McGraw-Hill series in electrical and computer engineering: Power and energy. McGraw-Hill, 2003.

[15] T. Gonen, "Modern power system analysis," CRC Press, 2016.

[16] O. D. Montoya-Giraldo, W. J. Gil-Gonzalez, and A. Garces-Ruız, "Flujo de potencia optimo para redes radiales y enmalladas empleando programacion semidefinida," *TecnoLogicas*, vol. 20, no. 40, pp. 29–42, Sep. 2017.

[17] S. H. Low, "Convex relaxation of optimal power flow Part I: Formulations and equivalence," *IEEE Trans. Control Netw. Syst*, vol. 1, no. 1, pp. 15–27, Mar. 2014, doi: 10.1109/TCNS.2014.2309732.

[18] M. Milovanovic, J. Radosavljevic, and B. Perovic, "A backward/forward sweep power flow method for harmonic polluted radial distribution systems with distributed generation units," *International Transactions on Electrical Energy Systems*, vol. 30, no. 5, Dec. 2019, doi: 10.1002/2050-7038.12310.

[19] A. Garces, "A linear three-phase load flow for power distribution systems," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 827–828, Jan. 2016, doi: 10.1109/TPWRS.2015.2394296.

[20] O. D. Montoya, J. S. Giraldo, L. F. Grisales-Nore˜na, H. R. Chamorro, and L. Alvarado-Barrios, "Accurate and efficient derivative-free three-phase power flow method for unbalanced distribution networks," *Computation*, vol. 9, no. 6, May. 2021, Art. No. 61, doi: 10.3390/computation9060061/.

[21] O. D. Montoya, L. F. Grisales-Norena, and W. Gil-Gonzalez, "Triangular matrix formulation for power flow analysis in radial DC resistive grids with CPLs," *IEEE Trans. Circuits Syst. II*, vol. 67, no. 6, pp. 1094–1098, Jun. 2020, doi: 10.1109/TCSII.2019.2927290.

[22] A. Garces, "Uniqueness of the power flow solutions in low voltage direct current grids," *Electr. Power Syst. Res*, vol. 151, pp. 149–153, Oct. 2017, doi: 10.1016/j.epsr.2017.05.031.

[23] O. D. Montoya, W. Gil-Gonzalez, and L. Grisales-Nore˜na, "An exact MINLP model for optimal location and sizing of DGs in distribution networks: A general algebraic modeling system approach," *Ain Shams Engineering Journal*, vol. 11, no. 2, pp. 409–418, Jun. 2020, doi: 10.1016/j.asej.2019.08.011.

[24] D. V. Tien, R. Gono, and Z. Leonowicz, "A new approach newton-raphson Load flow analysis in power system networks with STATCOM," in *Lecture Notes in Electrical Engineering. Springer International Publishing*, Apr. 2019, pp. 88–100, doi: 10.1007/978-3-030-14907-9-10.

[25] F. Milano, "Analogy and convergence of levenberg's and lyapunov-Based methods for power flow analysis," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1663–1664, Mar. 2016, doi: 10.1109/TPWRS.2015.2415455.